Progress of Theoretical Physics, Vol. 5, No. 6, November~December, 1950

An Algebraic Theory of the Density Matrix, I

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(Received May 10, 1950)

Introduction and summary

We state in the following a structure theory of the density matrices from the standpoint of second quantization. The difficulties involved in the quantum-mechanical many body problem have so far proved to be so tremendous that as yet some have been obliged to stop before exact treatments and others have retired to phenomenological model theories. There are two main tools to approach the problem, the theory of second quantization and the theory of density matrix. The former, in spite of its generality and compactness, has never revealed us its adaptability to the practical problems of quantum statistical mechanics. In fact, about the exchange effect, namely the statistical correlation of particles, the latter informs us the detailed features of it by application of the group theory, while the former lacks, at present, such an exposition.

For a long time a more general and simpler formulation than the current group theoretic method has been desired in the theoretical researches of spectroscopy. A distinguished contribution has been made by P. Jordan (1934)[1] and H. Ostertag (1937)[2] from the standpoint of second quantization. H. Ostertag made use of the theory of hypercomplex ring and introduced the original concept of the elementary algebra. (Part II.) Since the latter part of his formulation, however, seems to be complicated, it will be better, for clearness' sake, to avoid this part. Accordingly the concepts of the elementary algebra and the symmetric algebra will be introduced as a remedy for it. Then we shall complete our theory to include the whole theory of symmetric group. These concepts will serve to clarify and generalize the previous formulations and to unify the several group theoretic standpoints of the spectroscopy[3]. In § 1 and § 2 the representation of the quantized density of Bose or Fermi particles will reveal itself as the density matrix of Bose or Fermi particles or as the density matrix of many electrons with a certain term of spin multiplets. Indeed, the theory of the statistical operator, or the density matrix of J. von Neumann[4] and P.A.M. Dirac[5], the equilibrium properties of which have been fully examined by one of the authors (1940)[6], is the natural formalism of dealing with quantum gases. A contribution has been made by W. Kofink (1938)[7] to introduce the intrinsic degrees of freedom of particles. This attempt has resulted in a fair success.

His formulation consists in making use of the initial stages of the theory of spectroscopy, which will be reexamined in the second section and Part II. In § 3, as an example at zero temperature, the Hartree-Fock equation for a prescribed orbital and spin configuration will be derived, and in § 4 some properties of the reduced density matrix will be illustrated.

§ 1. Fermi statistics and Bose statistics

We consider a system composed of n identical Fermi or Bose particles immersed in a large thermostat at temperature T. The objective of our investigation is a Gibbs' virtual ensemble of identical systems which are canonically distributed with respect to energy. Every system is composed of n particles whose motion is determined by quantum mechanics.

The coordinates of the k-th particle in three, or with spin four, dimensional space are summarized in q_k. Every particle is described by the quantized wave function,

$$\varphi(q)=\sum_t a_t\psi_t(q)\,, \tag{1.1}$$

where $\psi_t(q)$ is orthogonal and normalized wave function belonging to the eigenvalue ε_t. The creation operator a^*, which is conjugate to the annihilation operator a, is to be considered as weighted by the Boltzmann factor $\exp(-\beta\varepsilon_t)$ and $\beta=\dfrac{1}{kT}$. The commutation relation is $[a_t, a_t^*]_\pm=\exp(-\beta\varepsilon_t)$, $[A, B]_\pm=AB\pm BA$ (+ for Fermi statistics, — for Bose statistics.)

It means

$$\rho(q, q|\beta)=[\varphi(q), \varphi(q)^*]_\pm=\sum_t \psi_t(q)\overline{\psi_t(q)}e^{-\beta\varepsilon_t}\,, \tag{1.2}$$

the unnormalized probability of existence of each particle at temperature T, and

$$f(\beta)=\int \rho(q, q|\beta)dq \tag{1.3}$$

is partition function.

An eigenvalue of n body problem is in Hartree field a sum of eigenvalues of individual particles. To an eigenvalue,

$$E=\sum_{i=1}^{n}\varepsilon_{t_i}\,, \tag{1.4}$$

belongs the eigenfunction $\Psi(q^n)=\det^{(\pm)}(\psi_{t_i}(q_k))$, where the coordinates of n particles are summarized in q^n and $\det^{(+)}$ means permanent for Bose particles. ($\det^{(-)}$ means determinant for Fermi particles.)

We define the density matrix of n particles by

$$\chi(\sum U_r\rho(q^n,q^{n\prime})U_r)=R^{(n)}(q^n, q^{n\prime})=\frac{1}{n!}\det{}^{(\pm)}(\rho(q_i, q_k'))\,, \tag{1.5}$$

$$\rho(q^n,q^{n'})=\frac{1}{n!}\varphi(q_1')^*\cdots\varphi(q_n')^*\varphi(q_n)\cdots\varphi(q_1)\,. \tag{1.6}$$

$\chi(\boldsymbol{A})$ means the result of taking trace of an operator $\boldsymbol{A}$, namely $\chi(\boldsymbol{A})=\sum_k A_{kk}$, $A_{kk}U_k=U_k\boldsymbol{A}U_k$. Here U_r is the so-called restriction operator, which selects the configuration $(r)=(r_1,\cdots\cdots,r_n)$ out of $\rho(q^n,q^{n'})$ and is as follows

$$U_r=N_{r_1}N_{r_2}\cdots N_{r_n}N^*_{r_{n+1}}\cdots,\ U_rU_r=U_r\,, \tag{1.7}$$

$$N_r=a_r^*a_r,\quad N_r^*=a_ra_r^*, \qquad \text{(Fermi statistics)}$$

$$U_r=N_{r_1,n_1}N_{r_2,n_2}\cdots N_{r_n,n_n}N^*_{r_{n+1}}\cdots\,, \tag{1.8}$$

(Bose statistics)

$$N_{r,n}=e_{r,nn}\,,\ N_r^*=N_{r,o}=e_{r,oo}\,,\ e_{r,nn'}\,e_{r,mm'}=e_{r,nm}\,\delta_{n'm'}\,.$$

N_k represents k-th state is occupied, $N_{k,m}$ represents k-th state is m-fold occupied, and N_k^* represents k-th state is vacant. And the operators a^*, a are the solution of the commutation relation $[a,a^*]_\pm=1$

$$a^*=\sum_{n=0}^{1}\sqrt{n}\,e_{nn-1}\,,\ a=\sum_{n=0}^{1}\sqrt{n}\,e_{n-1n}\,, \qquad \text{(Fermi statistics)} \tag{1.9}$$

$$a^*=\sum_{n=0}^{\infty}\sqrt{n}\,e_{nn-1}\,,\ a=\sum_{n=0}^{\infty}\sqrt{n}\,e_{n-1n}\,. \qquad \text{(Bose statistics)} \tag{1.10}$$

The notation π_r means a subgroup of the symmetric group π, under the influence of which the wave function $\psi_{r_1}(q)\cdots\psi_{r_n}(q_n)$ is invariant. The orbital configuration denoted by (r), to which this wave function belongs, is called type π_r. The notation $(r_1,\cdots,r_n)$ means $r_1,\cdots,r_n$ states are occupied and $(r)^*=(n_1,\cdots\cdots\cdots)$ means the first state is n_1-fold occupied, $\cdots$, n-th state is n_n-fold occupied and so on. And then we can readily show

$$\chi(U_r\rho(q^n,q^{n'})U_r)=\frac{n_1!\cdots n_n!}{n!}\sum_{\substack{P\\ \text{mod}}}\sum_{\substack{Q\\ \pi_r}}P\psi_{r_1}(q_1)\cdots\psi_{r_n}(q_n)Q\overline{\psi_{r_1}(q_1')}\cdots\overline{\psi_{r_n}(q_n')}\,. \tag{1.11}$$

For the simplicity the Boltzmann factors are omitted here. We understand

$$R^{(n)}(q^n)=R^{(n)}(q^n,q^n)=\sum_{(r)}|\psi_r(q^n)|^2e^{-\beta E_r}\,, \tag{1.12}$$

the probability at temperature T to find a particle at q_1, one at $q_2,\cdots$, one at q_n. Set (r) runs over all the allowable configurations.

After integration over n-k coordinates,

$$R^{(n)}(q^k)=R^{(n)}(q_1,\cdots,q_k)=\int R^{(n)}(q^n)dq_{k+1}\cdots dq_n \tag{1.13}$$

means the probability, in the presence of n particles, to find k particles at the prescribed positions $q_1q_2\cdots q_k$.

The partition function of n particles is defined by

$$f^{(n)}(\beta)=\int R^{(n)}(q^n)dq^n, \tag{1.14}$$

and, written down explicitly, runs as follows:

$$f_F^{(n)}(\beta)=\sum_{(\alpha)}\frac{(-1)^{\alpha_2+\alpha_4+\cdots}}{\alpha_1!\cdots\alpha_n!}\left(\frac{f(\beta)}{1}\right)^{\alpha_1}\cdots\left(\frac{f(n\beta)}{n}\right)^{\alpha_n}, \quad \text{(Fermi statistics)} \tag{1.15}$$

$$f_B^{(n)}(\beta)=\sum_{(\alpha)}\frac{1}{\alpha_1!\cdots\alpha_n!}\left(\frac{f(\beta)}{1}\right)^{\alpha_1}\cdots\left(\frac{f(n\beta)}{n}\right)^{\alpha_n}, \quad \text{(Bose statistics)} \tag{1.16}$$

$$1\alpha_1+\cdots+n\alpha_n=n\,. \tag{1.17}$$

One can readily find the relations

$$\frac{\partial f_F^{(n)}(\beta)}{\partial f(k\beta)}=(-1)^{k-1}\frac{1}{k}f_F^{(n-k)}(\beta)\,, \tag{1.18}$$

$$\frac{\partial f_F^{(n)}(\beta)}{\partial f(k\beta)}=\frac{1}{k}f_B^{(n-k)}(\beta)\,. \tag{1.19}$$

Renormalizing the commutation relation according to

$$[a_t, a_t^*]_\pm=e^{-\xi-\beta\varepsilon_t}\,, \tag{1.20}$$

and summing the quantized density operator over the different particle number,

$$\sum_{n=1}^{\infty}\rho(q^n, q^{n'})\,,$$

we get its trace for the case of one explicit coordinate as follows,

$$e^{-\xi}D(q, q'|\xi)=\sum_{n=1}^{\infty}e^{-n\xi}nR^{(n)}(q, q')\,, \tag{1.21}$$

$$D(\xi)=\sum_{n=1}^{\infty}e^{-n\xi}f^{(n)}(\beta)\,, \tag{1.22}$$

$$\sigma(qq'|\xi)=e^{-\xi}\frac{D(q, q'|\xi)}{D(\xi)}\,, \tag{1.23}$$

where $D(\xi)$ is the grand partition function and σ is the generalized density matrix introduced by one of the authors.

§ 2. Density matrices for n electron system

We start the following consideration of formulating the density matrix as the irreducible representation of the quantized density operator $\rho(q^n, q^{n'})$, before attacking mathematical problems which will fully be worked out in Part II.

The motion of k-th particle with intrinsic degrees of freedom is described by the quantized wave function

$$\varphi(q_k)=\sum_{t,\rho}a_{t\rho}\psi_t(x_k)\alpha\,(\sigma_k)\,, \tag{2.1}$$

where the three dimensional coordinates of the particle is summarized in x_k, and σ_k is the variable of intrinsic degrees of freedom. Suffix r indicates a translational state and suffix ρ indicates an intrinsic state of the particle. The summation is carried out over all the allowable states. The intrinsic state functions u_ρ are assumed to constitute an m-dimensional unitary space.

In the quantized density operator $\rho(q^n, q^{n'})$ there appear operators $a^*_{s\rho_1}\cdots a^*_{t\rho_n}$ $a_{t\rho'_n}\cdots a_{s\rho'_1}$. The structure of the algebra generated by these operators will be analyzed in Part II, and in this section we have interest in its results only. This algebraic formulation will serve to unify the several group theoretic standpoints of the theory of spectroscopy in a single scheme. According to the results of Part II, every simple ideal of the so-called symmetric algebra represents a certain term of spin multiplets.

Now we ask for an expression of the density matrix of n electrons, thus restricting ourselves to the case of two intrinsic degrees of freedom. The generalization[8] for the case of higher degrees of freedom will readily be achieved and applied to the spectroscopy of nucleus.

The density matrix corresponding to a prescribed spin multiplets S is defined by,

$$\rho_S(q^n, q^{n'}) = \sum_{(r)} \chi_{r,S}(U_r \rho(q^n, q^{n'}) U_r) , \tag{2.2}$$

where U_r is the restriction operator selecting a prescribed orbital configuration (r) and $\chi_{r,S}(\boldsymbol{A})$ means in general the result of taking trace of an operator $\boldsymbol{A}$ with respect to the simple ideal $I_{r,S}$ attached to configuration (r) and total spin quantum number S.

Let the eigenfunction corresponding to an eigenvalue E_r be $\psi_r(x_1,\cdots, x_n)$, and after the experience of the restriction process and the separation of the intrinsic states, (2.1) becomes

$$\rho_S(x_1,\cdots, x_n ; x_1',\cdots, x_n') = \frac{1}{n!} \sum_{(r)} \sum_{P} \sum_{Q} \frac{1}{g_r} \chi_{r,S}(\boldsymbol{P}_r \boldsymbol{Q}_r^{-1}) \psi_r(P ; x_1,\cdots, x_n) \cdot \overline{\psi_r(Q ; x_1',\cdots, x_n')}, \tag{2.3}$$

$$\boldsymbol{P}_r = \sum a^*_{1\rho_1}\cdots a^*_{n\rho_n} a_{t\rho_n}\cdots a_{s\rho_1} , \quad P = \begin{bmatrix} s \cdots t \\ 1 \cdots n \end{bmatrix} ,$$

where g_r is the order of the subgroup π_r, and the summation is carried out over all the group elements of the symmetric group π. The configuration (r), the wave function $\psi_r(x_1,\cdots, x_n)$ and the operator $\boldsymbol{S}_r$ are called to belong to type π_r. Making use of the fact that the trace of the operator $\boldsymbol{S}_r$ belonging to type π_r is a linear combination of traces of the operator $\boldsymbol{S}_0$ belonging to type unity, namely to an orbital configuration $(r)^* = (1, 1\cdots, 1, 0,\cdots)$, we get

$$\rho_S(x_1,\cdots, x_n ; x_1',\cdots, x_n') = \frac{1}{n!} \sum_{P} \chi_S(\boldsymbol{P}_0) \rho(x_1, x'_{i_1}|\beta) \cdots \rho(x_n, x'_{i_n}|\beta) , \tag{2.4}$$

where

$$P=\begin{bmatrix} 1\cdots n \\ i_1\cdots i_n \end{bmatrix}.$$

Integrating $\rho_S(x_1,\cdots,x_n;x_1,\cdots,x_n)$ over $3n$ spatial coordinates, we get

$$f_S^{(n)}(\beta)=\sum_{(\alpha)}\frac{\chi_S^{(\alpha)}}{\alpha_1!\cdots\alpha_n!}\left(\frac{f(\beta)}{1}\right)^{\alpha_1}\cdots\left(\frac{f(n\beta)}{n}\right)^{\alpha_n}, \tag{2.5}$$

and $f_S^{(n)}(\beta)$ is the partition function of n particles. Partition $1\alpha_1+\cdots+n\alpha_n=n$ determines a class of the symmetric group and $\chi_S^{(\alpha)}=\chi_S(R_0)$. R_0 is in the class (α). $\chi_S(\boldsymbol{R}_0)$ is given by the difference of two traces of symmetric algebra taken with respect to the two algebras of magnetic quantum number M_z as follows

$$\chi_S(\boldsymbol{R}_0)=\chi_{|M_z|=S}(\boldsymbol{R}_0)-\chi_{|M_z|=S+1}(\boldsymbol{R}_0)\,. \tag{2.6}$$

The traces of any element of symmetric algebra are equivalent to the traces of the associate representations of the corresponding element of symmetric group. It has been pointed by W. Kofink that $f_S^{(n)}(\beta)$ is a homogeneous function of n-th order of variable $f(\beta),f^{\frac{1}{2}}(2\beta),\cdots,f^{\frac{1}{n}}(n\beta)$, and the reduced density matrix is obtained through $f_s^{(n)}(\beta)$ making use of Euler's theorem for homogeneous functions. If one notices that (2.5) has the alternative form,

$$f_S^{(n)}(\beta)=\begin{vmatrix} f_F^{(\frac{n}{2}+S)}(\beta) & f_F^{(\frac{n}{2}+S+1)}(\beta) \\ f_F^{(\frac{n}{2}-S-1)}(\beta) & f_F^{(\frac{n}{2}-S)}(\beta) \end{vmatrix}, \tag{2.7}$$

one can prove the relation

$$\frac{\partial f_S^{(n)}(\beta)}{\partial f(k\beta)}=\frac{(-1)^{k-1}}{k}\left(f_{S-\frac{k}{2}}^{(n-k)}(\beta)+f_{S+\frac{k}{2}}^{(n-k)}(\beta)\right), \tag{2.8}$$

from (1.18)

One can also find the partition function belonging to a certain spin magnetic quantum number (" Gesamtspin " of Kofink) $M_z\geqq 0$ is

$$f_{M_z}^{(n)}(\beta)=\sum_{S\geqq M_z}^{\frac{n}{2}}f_S^{(n)}(\beta)=f_F^{(\frac{n}{2}-M_z)}(\beta)\,f_F^{(\frac{n}{2}+M_z)}(\beta)\,. \tag{2.9}$$

It represents a simple evidence for the fact that, statistically, antiparallel spins do not correlate each other.

§ 3. The Hartree-Fock approximation

It is proposed here to derive the Hartree-Fock equation from a more general standpoint than usual.

A system of n Fermi particles with n self-energies K and $\binom{n}{2}$ binary interaction energies V will be considered. For the total energy of the form,

$$H(q_1\cdots q_n, q_1'\cdots q_n')=\sum_{i=1}^{n}K(q_i, q_i')+\sum_{i>k}^{n}V(q_i\, q_k, q_i'\, q_k'), \tag{3.1}$$

the Hamilton operator in the configuration space is given by

$$\boldsymbol{H}=\int K(q,q')\rho^{(n)}(q',q)dq'dq+\frac{1}{2}\int V(qr, q'r')\rho^{(n)}(q'r', qr)dq'dr'dqdr. \tag{3.2}$$

If we have interest in the micro-canonical ensemble at temperature zero, it is sufficient to consider the energy operator in the form

$$\boldsymbol{H}=\int\psi^*(q')K(q', q)\psi(q)dq'dq+\frac{1}{2}\int\psi^*(q')\psi^*(r')V(q'r', qr)\psi(r)\psi(q)dq'dr'dqdr, \tag{3.3}$$

or more explicitly in the form,

$$\boldsymbol{H}=\sum_{t=1}^{\infty}\sum_{\rho=0}^{1}\langle K\rangle_t N_{t\rho}+\frac{1}{2}\sum_{\substack{s=1\\ s'=1}}^{\infty}\sum_{\substack{t=1\\ t'=1}}^{\infty}\sum_{\rho=0}^{1}\sum_{\rho'=0}^{1}\langle V\rangle_{st,t's'}a_{s\rho}^*a_{t\rho'}^*a_{s'\rho'}a_{t'\rho}, \tag{3.4}$$

where the total energy H does not include spin variables.

When it is assumed that m orbital states are doubly occupied and $(n-2m)$ orbital states are singly occupied, this configuration is determined by the restriction operator

$$U_r=N_1(2)\cdots N_m(2)N_{m+1}(1)\cdots N_n(1)N_{n+1}(0)\cdots, \tag{3.5}$$

and

$$N_t(2)=N_{t1}N_{t0},\ N_t(1)=N_{t1}(1-N_{t0})+N_{t0}(1-N_{t1}),\ N_t(0)=(1-N_{t1})(1-N_{t0}). \tag{3.6}$$

After selecting this configuration out of the energy operator we get

$$\begin{aligned}U_r\boldsymbol{H}U_r=\Big[&\sum_{t=1}^{m}\langle K\rangle_t+\sum_{t=1}^{n}\langle K\rangle_t+\sum_{s=1}^{m}\sum_{t=1}^{n}\{2\langle V\rangle_{st,st}-\langle V\rangle_{st,ts}\}\\&+\frac{1}{2}\sum_{s=m+1}^{n}\sum_{t=m+1}^{n}\{\langle V\rangle_{st,st}+(st)_0\langle V\rangle_{st,ts}\}\Big]U_r,\end{aligned} \tag{3.7}$$

where $(st)_0$ is an element of symmetric algebra attached to $(n-2m)$ singly occupied orbital states and given in the form

$$(st)_0=(-N_{s1}N_{t1}-N_{s0}N_{t0}+a_{s1}^*a_{t0}^*a_{s0}a_{t1}+a_{s0}^*a_{t1}^*a_{s1}a_{t0})U_r. \tag{3.8}$$

It will be better to introduce the Dirac's density matrices at zero temperature

$$\begin{aligned}\tau_1(x, x')&=\sum_{t=1}^{m}\psi_t(x)\overline{\psi_t(x')},\\ \tau_2(x, x')&=\sum_{t=1}^{n}\psi_t(x)\overline{\psi_t(x')},\end{aligned} \tag{3.9}$$

and to denote

$$
\begin{aligned}
\sum_{t=1}^{m}\langle K\rangle_t &=\int K(x,x')\tau_1(x',x)dx'dx\,,\\
\sum_{t=1}^{n}\langle K\rangle_t &=\int K(x,x')\tau_2(x',x)dx'dx\,,\\
\sum_{s=1}^{m}\sum_{t=1}^{m}\langle V\rangle_{st,st} &=\int V(xy,x'y')\tau_1(y,y')\tau_1(x,x')dx'dy'dxdy\,,\\
\sum_{s=1}^{n}\sum_{t=1}^{n}\langle V\rangle_{st,st} &=\int V(xy,x'y')\tau_2(y',y)\tau_2(x',x)dx'dy'dxdy\,,\\
\sum_{s=1}^{m}\sum_{t=1}^{m}\langle V\rangle_{st,ts} &=\int V(xy,y'x')\tau_1(x',x)\tau_1(y',y)dx'dy'dxdy\,,\\
\sum_{s=1}^{n}\sum_{t=1}^{n}\langle V\rangle_{st,ts} &=\int V(xy,y'x')\tau_2(x',x)\tau_2(y',y)dx'dy'dxdy\,.
\end{aligned}
\tag{3.10}
$$

Taking trace with respect to a certain magnetic or total spin quantum number, we get for the mean energy value

$$
\begin{aligned}
\langle \boldsymbol{H}\rangle_{AV}=&K(x,x')\{\tau_1(x',x)+\tau_2(x',x)\}dx'dx\\
&+\frac{1}{2}\int V(xy,x'y')\{\tau_1(y',y)+\tau_2(y',y)\}\{\tau_1(x',x)+\tau_2(x',x)\}dx'dy'dxdy\\
&-\frac{1}{2}\int V(xy,y'x')\{\tau_1(x',x)\tau_2(y',y)+\tau_1(y',y)\tau_2(x',x)\}dx'dy'dxdy\\
&+\frac{1}{2}\chi((12)_0)\int V(xy,y'x')\{\tau_2(x',x)-\tau_1(x',x)\}\{\tau_2(y',y)-\tau_1(y',y)\}\\
&\times dx'dy'dxdy\,.
\end{aligned}
\tag{3.11}
$$

Under the following conditions,

$$
\begin{aligned}
&\tau_1\cdot\tau_1=\tau_1,\ \tau_2\cdot\tau_2=\tau_2,\ \tau_1\cdot\tau_2=\tau_2\cdot\tau_1=\tau_1,\\
&\langle\delta\tau_1\rangle_{AV}=0,\ \langle\delta\tau_2\rangle_{AV}=0,
\end{aligned}
\tag{3.12}
$$

and with the Lagrange multipliers,

$$
\begin{aligned}
&W_1(\text{matrix order } m^2),\quad W_2(\text{matrix order } n^2),\\
&\omega_1\,,\qquad\qquad \omega_2\,,
\end{aligned}
\tag{3.13}
$$

the postulate of minimum energy,

$$\delta\langle \boldsymbol{H}\rangle_{AV}=0\,, \tag{3.14}$$

leads to the following two equations

$$
\begin{aligned}
&2\int\{K(x,x')+V(x,x')\}\psi_i(x')dx'-\int\Big\{\sum_{k=1}^{m}2F_{ki}(x,x')+\sum_{k=m+1}^{n}F_{ki}(x,x')\Big\}\psi_k(x')dx'\\
&=(\omega_1+\omega_2)\psi_i(x)+\sum_{k=1}^{m}\psi_k(x)W_{1,ki}+\sum_{k=1}^{n}\psi_k(x)W_{2,ki}\,,
\end{aligned}
\tag{3.15}
$$

for $i=1,\cdots,m$ and

$$\int\{K(x,x')+V(x,x')\}\psi_i(x')dx'$$
$$-\int\left\{\sum_{k=1}^{m}F_{ki}(x,x')\psi_k(x')-\sum_{k=m+1}^{n}\chi((12)_0)F_{ki}(x,x')\psi_k(x')\right\}dx'$$
$$=\omega_2\psi_i(x)+\sum_{k=1}^{n}\psi_k(x)W_{2,ki}\ , \tag{3.16}$$

for $i=m+1,\cdots,n$, where

$$V(x,x')=\int V(xy,x'y')\{\tau_1(y',y)+\tau_2(y',y)\}dy'dy, \tag{3.17}$$

$$F_{ki}(x,x')=\int\overline{\psi_k(y)}\,V(xy,x'y')\psi_i(y')dy'dy\ . \tag{3.18}$$

If $\chi((12)_0)$, the character of the exchange operator, is replaced by -1, the two equations (3.15)(3.16) take the same form as those derived by V. Fock.[9] It is valid when the magnetic quantum number M_z or total spin quantum number S is $\frac{n}{2}-m$. For the general cases $\chi((12)_0)$ is to be replaced by

$$-\frac{1}{2}\frac{(n-2m)^2-2(n-2m)+4M_z^2}{(n-2m)(n-2m-1)}, \tag{3.19}$$

or

$$-\frac{1}{2}\frac{(n-2m)^2-4(n-2m)+4S^2+4S}{(n-2m)(n-2m-1)}. \tag{3.20}$$

The matrix elements of the Lagrange multipliers are given by

$$W_{1,ki}+\omega_1\delta_{ki}=[K(x,x')+V(x,x')-T_1(x,x')]_{ki}\,,\qquad i,k=1,\cdots,m\,, \tag{3.21}$$

$$W_{2,ki}+\omega_2\delta_{ki}=[K(x,x')+V(x,x')-T_2(x,x')]_{ki}\,,\qquad i,\ k=1,\cdots,m, \tag{3.22}$$

$$W_{2,ki}+\omega_2\delta_{ki}=[K(x,x')+V(x,x')-T_1(x,x')+\chi((12)_0)\{T_2(x,x')-T_1(x,x')\}]_{ki}\,,$$
$$i \text{ or } k=m+1,\cdots,n,\ k \text{ or } i=1,\cdots,n, \tag{3.23}$$

where

$$T_1(x,x')=\int V(xy,y'x')\tau_1(y',y)dy'dy, \tag{3.24}$$

$$T_2(x,x')=\int V(xy,y'x')\tau_2(y',y)dy'dy. \tag{3.25}$$

The Thomas-Fermi approximation will readily be worked out.[10]

§ 4. Reduced density matrices

The commutation relation of quantized wave function enables us to obtain the formulae of the reduced density matrix. The following recursion formula is derived from the quantized density operator of n electrons by transferring the creation operators to the right and the annihilation operators to the left[11]

$$n\rho^{(n)}(q^n, q^{n'}) = \rho^{(n-1)}(q^{n-1}, q^{n-1'})\rho^{(1)}(q_1, q_1')$$
$$-\sum_j \rho^{(n-1)}(q_1\cdots q_{j-1}, q_{j+1}\cdots q_n ; q_j'q_2'\cdots q_n')\rho(q_j, q_1'). \qquad (4.1)$$

Integrating this formula over the coordinates $q_2, \cdots, q_n$ (with spin), one gets

$$n\rho^{(n)}(q_1, q_1') = f_Q^{(n-1)}(\beta)\rho^{(1)}(q_1', q_1) - (n-1)\int \rho^{(n-1)}(q_1, r)\rho(r, q_1')dr. \qquad (4.2)$$

One gets further by successive applications of the formula,

$$n\rho^{(n)}(q_1, q_1') = \sum_{k=1}^{n}(-1)^{k-1}f_Q^{(n-k)}(\beta)\rho^{(1)}(q_1, q_1'|k\beta) \qquad (4.3)$$

separating the Boltzmann factor from a^* :

$$\rho^{(1)}(q_1, q_1'|k\beta) = \sum_s\sum_t a_s^* a_t \overline{\psi_s(q_1')}\psi_t(q_1)\varepsilon^{-k\beta\varepsilon_s}.$$

One can find

$$\chi_s(f_Q^{(n-k)}(\beta)\cdot\rho^{(1)}(q_1, q_1'|k\beta)) = k\frac{\partial f_s^{(n)}(\beta)}{\partial f(k\beta)}\rho(q_1, q_1'|k\beta) \qquad (4.4)$$

and finally the reduced density matrix is obtained from (2.8)

$$R_s^{(n)}(q_1, q_1') = \frac{1}{n}\Bigg[\sum_{k=1}^{\frac{n}{2}-s}(-1)^{k-1}\Big\{\rho(q_1, q_1'|k\beta)f_{s+\frac{k}{2}}^{(n-\frac{k}{2})}(\beta) + (-1)^{2S}\rho(q_1, q_1'|(2S+k+1)\beta)$$
$$f_{\frac{k}{2}-\frac{1}{2}}^{(n-2S-k-1)}(\beta)\Big\} + \sum_{k=1}^{2S}(-1)^{k-1}\rho(q_1, q_1'|k\beta)f_{S-\frac{k}{2}}^{(n-\frac{k}{2})}(\beta)\Bigg]. \qquad (4.5)$$

The reduced density matrix belonging to magnetic quantum number M_z is also obtained from (2.9)

$$R_{M_z}^{(n)}(q_1, q_1') = \frac{1}{n}\sum_k(-1)^{k-1}\rho(q_1, q_1'|k\beta)\Big\{f_F^{(\frac{n}{2}+M_z-k)}(\beta)f_F^{(\frac{n}{2}-M_z)}(\beta)$$
$$+ f_F^{(\frac{n}{2}+M_z)}(\beta)f_F^{(\frac{n}{2}-M_z-k)}(\beta)\Big\}$$
$$= \frac{\left(\frac{n}{2}+M_z\right)}{n}R^{(\frac{n}{2}+M_z)}(q_1, q_1')f_F^{(\frac{n}{2}-M_z)}(\beta) + \frac{\left(\frac{n}{2}-M_z\right)}{n}R^{(\frac{n}{2}-M_z)}(q_1, q_1')f_F^{(\frac{n}{2}+M_z)}(\beta)$$
$$(4.6)$$

References

1) P. Jordan, ZS. f. Phys. **91** (1934), 284.
2) H. Ostertag, ZS. f. Phys. **106** (1937), 329.
3) H. Weyl, *Gruppentheorie und Quantenmechanik* (1931).
E. Wigner, *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren* (1931).
B. L. van der Waerden, *Die Gruppentheoretische Methode in der Quantenmechanik* (1932).
4) J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (1932).
5) P. A. M. Dirac, *The Principles of Quantum Mechanics* (1935).
6) K. Husimi, Proc. Phys. Math. Soc. Japan **22** (1940), 264.
7) W. Kofink, Ann. d. Phys. **28** (1937), 264.
8) T. Nishiyama, Prog. Theor. Phys. **5** (1950), 135.
9) V. Fock, ZS. f. Phys. **61** (1930), 126.
10) H. Ito, Lecture at the symposium of the Phys. Soc. Japan (1948).
11) For example, $2\rho^{(2)}(q_1, q_2; q_1', q_2') = \rho^{(1)}(q_2, q_2')\, \rho^{(1)}(q_1, q_1') - \rho^{(1)}(q_1, q_2')\, \rho(q_2, q_1')$.

Note added in proof

Recently Prof. Tomonaga (Prog. Theor. Phys. **5** (1950), 544) has made a distinguished contribution to the many fermion problems by applying the theory of sound quanta. We, independently, have proposed a similar consideration to describe fermion and boson assemblies in terms of the density and the current operators represented by the Fourier coefficients $\varphi^*(k)$, $\varphi(k)$ of the wave functions: $\rho(k) = \sum_l \varphi^*_{l-\frac{k}{2}} \varphi_{l+\frac{k}{2}}$, $j(k) = \sum_l l \varphi^*_{l-\frac{k}{2}} \varphi_{l+\frac{k}{2}}$, which are subjected to the commutation relations,

$$[\rho(k), \rho(l)] = 0, \qquad [\rho(k), j(l)] = \frac{\hbar}{m} k\rho_{k+l},$$

$$[j(k)^x, j(l)^y] = \frac{\hbar}{m} (j(k+l)^x k^y - j(k+l)^y l^x),$$

where x represents the x-components of vectors. So far as the one dimensional problem is concerned, our results obtained are verified from Tomonaga's point of view. Introducing spin variables the density operators of $(+)$ spin and $(-)$ spin are denoted by $\rho_0(k)$ and $\rho_1(k)$ respectively.

Under Tomonaga's approximation the canonically conjugate quantities exist and they are denoted by $\pi_0(k)$ and $\pi_1(k)$. For the one dimensional eigenvalue problem it is convenient to take the quantities $(\rho_0(k)+\rho_1(k))/\sqrt{2}$, $(\pi_0(k)+\pi_1(k))/\sqrt{2}$, $(\rho_0(k)-\rho_1(k))/\sqrt{2}$, $(\pi_0(k)-\pi_1(k))/\sqrt{2}$ as the canonical variables.

Progress of Theoretical Physics, Vol. 5 No. 6, November~December, 1950

Remarks on Bethe's Neutral Theory of Nuclear Force

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(Received May 11, 1950)

§ 1. Introduction

The static nuclear potential derived from current meson theory does not permit any stable nuclear system because of its strong singularity at the origin. The existence of deuteron, however, shows that there must be some reasonable nuclear potential which has admissible singularity at the origin and explains many empirically known properties of deuteron and also of other nuclear systems. According to the above opinions many authors have repeated their calculations based on respectively different grounds. Some are based upon purely phenomenological grounds, without regard to any special consequences of particular type of meson theory, only some general consequences being considered. For example, Rarita and Schwinger[1] obtained agreeable results using square well potential with depth and width so chosen as to fit some empirical data. But from those phenomenological calculations nothing decisive about current meson theory can be concluded. If we want to say something concerning the type of meson field and how to treat potentials mathematically, we must have recourse to other ways which more or less are based upon meson theory.

For this purpose when we want to start directly from the static nuclear potentials derived from current meson theory, we have to devise some measures to avoid the inadmissibe singularity. Thus two methods have been proposed, one of which is the so-calld cut-off method and the other is that of cancellation of singularity by mixing two types of meson fields appropriately. Following the latter method, which is originated from Schwinger,[2] Jauch and Hu[3] and Wu and Foley,[4] using Møller-Rosenfeld-Schwinger mixture, calculated deuteron problems and found that with adjustable constants appropriately chosen in order to fit some empirical data the value of the quadrupole moment of the deuteron is always too small to fit the empirical one. Also the author examined the non-symmetrical mixture of charged and neutral vector meson fields.[5] In this case the unfavorable singularity of the tensor force is eliminated just as the case of Møller-Rosenfeld-Schwinger mixture. The results of the calculation was the same as above. From these it is concluded that the procedure of eliminating the singularity of tensor force by cancelling the contributions of two types of meson fields makes the important tensor force so weak that the value of the quadrupole moment becomes

very small compared with the experimental one and such a procedure is therefore not an appropriate method in researching the deuteron problems.

With regard to the cut-off method, Bethe[6] obtained satisfactory results concerning the deuteron problem with neutral vector meson theory. This cut-off method is naturally unsatisfactory but there was no other ingenious way except doing so. Indeed this cut-off potential has been hoped to be a good approximation to the true one if it should be right to consider the potential having a definite mathematical expression. This opinion was supported by the proof given by Schwinger[7] that the detailed radial dependence of the potential near the origin does not play any important role. Thus, because of satisfactory results obtained by Bethe's calculation, the expression of the nuclear potential derived from neutral vector meson theory was believed to be agreeable in deuteron problems, although we cannot accept the concept of nuclear interaction by virtue of neutral mesons.

Recently many authors[8] showed that the nuclear potential given by meson theory does not behave so abnormally near the origin as shown before but has, in fact, perfectly regular singularity. But its mathematical expression is very complicated and not suitable for the detailed analysis of nuclear problems. At the same time it was shown that the behavior far from the origin is correctly represented by the previous expression.

Accordingly, as another method avoiding the inadmissible singularity, we tried the way to modify the previous expression so as to behave normally at the origin and to give the same value far from the origin. The most simple and reasonable way to realize the above plan is to replace the irregular potential

$$\frac{e^{-\varkappa r}}{r}\left(\frac{1}{3}+\frac{1}{\varkappa r}+\frac{1}{\varkappa^2 r^2}\right)\left(\frac{3(\sigma_1\cdot r)(\sigma_2\cdot r)}{r^2}-\sigma_1\cdot\sigma_2\right)$$

by the simple Yukawa type potential

$$\frac{1}{3}N\frac{e^{-\xi\varkappa r}}{r}\left(\frac{3(\sigma_1\cdot r)(\sigma_2\cdot r)}{r^2}-\sigma_1\cdot\sigma_2\right)$$

where two adjustable parameters N and ξ are introduced. If we choose $N=\xi=1$, the new potential is obtained by simply omitting the terms having higher singularities than the second in the old one and evidently behaves in the same manner as the old one in the distant places from the origin. We have introduced ξ for adjusting the range of the tensor force and N for adjusting the strength of the tensor force. Because we have some reasons mentioned above to believe that this approximation method is more reasonable from meson theoretical grounds than the other two ways, cut-off method and cancellation method, we tried the present work in order to say definitely which potential is the most agreeable among the many potentials derived from current meson theory and what kind of potential do really reproduce the empirical data concerning the deuteron and other low energy nucleon interactions. In order to answer the above questions

it is unsatisfactory to stand on purely phenomenological grounds and we must start from meson theoretical ground as much as possible. But it must be borne in mind that the consequences obtained from the present work have meaning only if this approximation method is the most reasonable one in the present status of meson theory, at least so believes the author. If other approximation method should be more reasonable, the consequences written below would have to be abandoned.

§ 2. Method for calculation

We carried out the calculation after the example of Wu and Foley.[4] Only in some respects we made modifications in order to simplify the calculations. We shall show its outline in what follows.

1. Empirical data:

a. The binding energy of the deuteron is (2.208 ± 0.007) Mev.[9]

b. The total cross section of the scattering of slow neutrons by protons is $(20.36 \pm 0.10) \times 10^{-24}$cm.[9) 10)]

c. The effective range r_0 for ^3S proton-neutron scattering is estimated as $(1.56 \pm 0.13) \times 10^{-13}$cm.[9)11)]

d. The effective range r_0 for ^1S proton-neutron scattering cannot be well determined at present, but may be smaller than that for ^1S proton-proton interaction;[11] $(2.71 \pm 0.13) \times 10^{-13}$cm.[9]

e. The quadrupole moment of the deuteron is found to be 2.73×10^{-27}cm.[2 12)]

f. D-state probability of the deuteron is estimated as 3%[13] from the magnetic moment of the deuteron.

g. The mass of π-meson was assumed to be $286 m_e$.*

2. Nuclear potentials:

If we make the modification mentioned in the introduction for the irregular tensor force, the expression for the nuclear potentials are given by the following formulae for p-p and p-n systems:

$$\begin{cases} {}^1S \quad ; \; V(x) = (a - 2b) e^{-x}/x, & (1) \\ {}^3S + {}^3D \, ; \; V(x) = \left(a + \dfrac{2}{3} b\right) e^{-x}/x - \dfrac{b}{3} N(e^{-\xi x}/x) \Lambda & (2) \end{cases}$$

for neutral vector meson field, and

$$\begin{cases} {}^1S \quad ; \; V(x) = -(1/2) c e^{-x}/x, & (3) \\ {}^3S + {}^3D \, ; \; V(x) = -(1/2) c e^{-x}/x - (1/2) c N(e^{-\xi x}/x) \cdot \Lambda & (4) \end{cases}$$

* The most recent experimental value, 276 m_e, was known to me after the calculations, but the main conclusions will be conserved even if we make this alteration.

for symmetrical pseudoscalar meson field, where

$$\Lambda=\frac{3(\sigma_1\cdot x)(\sigma_2\cdot x)}{x^2}-\sigma_1\cdot\sigma_2, \tag{5}$$

$$a=\frac{g_1^2}{\hbar c}\frac{M}{m_\pi},\quad b=\frac{g_2^2}{\hbar c}\frac{M}{m_\pi},\quad c=\frac{f_2^2}{\hbar c}\frac{M}{m_\pi}, \tag{6}$$

$$x=\varkappa r,\quad \varkappa=\frac{m_\pi c}{\hbar} \tag{7}$$

and M and m_π are the masses of nucleon and π-meson and g_1, g_2 and f_2 are vector, tensor and pseudovector coupling constants of vector and pseudoscalar meson fields respectively.

3. Slow neutron scattering:

The Schrödinger equation of the ^1S wave is

$$\frac{d^2u}{dx^2}+\left(k^2+(2b-a)\frac{e^{-x}}{x}\right)u=0, \tag{8}$$

where

$$k^2=ME/\hbar^2\varkappa^2, \tag{9}$$

E being the energy of the incident neutron in the center of mass system. The equation (8) is written for neutral vector case, but for symmetrical pseudoscalar meson field we have only to replace $(2b-a)$ by $c/2$ and in all the following formulae the discrepancies between two fields are trivial like this. So we shall give the formulae only for vector case below and for symmetrical pseudoscalar case only the results are given later.

The observed total cross section for slow neutron is

$$\sigma=\frac{3}{4}\sigma_t+\frac{1}{4}\sigma_s=20.36\times10^{-24}\text{cm}^2.$$

The contribution of singlet state scattering σ_s to σ is known from the scattering of slow neutrons by parahydrogen[14] and is given by

$$\sigma_s=73.7\times10^{-24}\text{cm}^2*. \tag{10}$$

As

$$\sigma_s\cong\frac{4\pi\sin^2\delta_0}{k_0^2},\quad k_0^2=\frac{ME}{\hbar^2},$$

* This value is the one which is used in Wu and Foley's paper, see reference 4. We also used this value following them, but the correct value is somewhat smaller, namely 70.88 barns, according to Bethe, see reference 9. If we take this correct value, we obtain $k/\sin\delta_0=0.0567$ instead of 0.0556 in equation (11). However this modification induces very little change in the value of $(2b-a)$ in (15) as can easily be seen from (12′) and (14′) and the main conclusions of this paper are certainly not changed by this correction.

one obtains, for $m_\pi=286m_e$,

$$\frac{k}{\sin\delta_0}=0.0556. \tag{11}$$

In order to determine the value of $(2b-a)$ which gives the observed phase δ_0 (11), we shall employ the variational equation

$$\delta\int_0^\infty u\left[\frac{d^2}{dx^2}+k^2+(2b-a)\frac{e^{-x}}{x}\right]u\,dx=0, \tag{12}$$

which is equivalent to (8). We assume the trial wave function[15]

$$u=\cos\delta_0\cdot\sin kx+\sin\delta_0(1-e^{-x})(1+\beta e^{-x})\cos kx, \tag{13}$$

where β is a variational parameter. Further we make use of the following identity from the theory of scattering

$$\sin\delta_0=\frac{1}{k}\int_0^\infty\sin kx\left[(2b-a)\frac{e^{-x}}{x}\right]u\,dx. \tag{14}$$

Upon substituting (13) into (12), (14) and neglecting higher order terms in k which is of the order 10^{-4} for the energy of the order of 1 ev. from (9), one obtains

$$[1-6(2b-a)ln(16/15)]\beta=-1+6(2b-a)[ln(9/8)+k\cot\delta_0/6], \tag{12'}$$

$$[(2b-a)/6]\beta=1-(2b-a)[(1/2)+k\cot\delta_0]. \tag{14'}$$

From the above two equations $(2b-a)$ and β are both determined and are given:

$$2b-a=1.5676, \qquad \beta=0.4955. \tag{15}$$

4. Ground state wave function of the deuteron is $[u+(1/2\sqrt{2})Aw]\chi_1^M$, where χ_1^M is the symmetrical spin wave function and u and w are the ^3S and ^3D wave functions in the L,S representation. The equations for u and w are

$$\left\{\begin{aligned}&\frac{d^2u}{dx^2}+\left[-\alpha^2-\left(a+\frac{2}{3}b\right)\frac{e^{-x}}{x}\right]u=-2^{\frac{3}{2}}\frac{b}{3}N\frac{e^{-\xi x}}{x}w\\ &\frac{d^2w}{dx^2}+\left[-\alpha^2-\left(a+\frac{2}{3}b\right)\frac{e^{-x}}{x}-\frac{2}{3}bN\frac{e^{-\xi x}}{x}-\frac{6}{x^2}\right]w\\ &\qquad\qquad=-2^{\frac{3}{2}}\frac{b}{3}N\frac{e^{-\xi x}}{x}u\end{aligned}\right. \tag{16}$$

where α^2 is related to the absolute value of the binding energy $\epsilon=2.208$ Mev. by

$$\alpha^2=\frac{M\epsilon}{\hbar^2\varkappa^2}=0.09654. \tag{17}$$

First we tried to determine a and b respectively from equation (16), choosing specially $N=\xi=1$. Next we took $a=0$ according to Bethe's single force

hypothesis in order to examine the appropriateness of Bethe's neutral vector meson field on the basis of the approximation method mentioned in the introduction, which is a main subject of this paper. In the case of symmetrical pseudoscalar meson field, force constant c is determined from slow neutron scattering just as in the case of single force hypothesis. In these cases, we tried to determine N so as to satisfy (16) for appropriately chosen ξ values. All these calculations are carried out almost in the same line and in the following its outline is shown for the case of single force hypothesis of neutral vector meson field. Simply writing $2b \equiv m = 1.5676$ and transposing all the terms containing undetermined parameter N to the right hand side, we can rewrite (16) by

$$\left\{\begin{aligned} &\frac{d^2u}{dx^2}+\left[-a^2-\frac{m}{3}\frac{e^{-x}}{x}\right]u=-\frac{\sqrt{2}}{3}mN\frac{e^{-\xi x}}{x}w,\\ &\frac{d^2w}{dx^2}+\left[-a^2-\frac{6}{x^2}-\frac{m}{3}\frac{e^{-x}}{x}\right]w=\left[\frac{1}{3}m\frac{e^{-\xi x}}{x}w-\frac{\sqrt{2}}{3}m\frac{e^{-\xi x}}{x}u\right]N. \end{aligned}\right. \tag{16'}$$

On denoting I_1 and I_2 by

$$I_1=\int_0^\infty\left[u'^2+w'^2+a^2(u^2+w^2)+\frac{6}{x^2}w^2+\frac{m}{3}\frac{e^{-x}}{x}(u^2+w^2)\right]dx, \tag{18}$$

$$I_2=\int_0^\infty\left[\frac{2\sqrt{2}}{3}m\frac{e^{-\xi x}}{x}uw-\frac{m}{3}\frac{e^{-\xi x}}{x}w^2\right]dx, \tag{19}$$

it is seen that the problem is equivalent to the variational problem

$$\delta(I_1/I_2)=0 \tag{20}$$

and the eigenvalue N is the minimum value of

$$N=\frac{I_1}{I_2}. \tag{21}$$

We employed in the calculation the trial wave functions*

$$\left\{\begin{aligned} &u(x)=(1-e^{-\beta x})e^{-\alpha x},\\ &w(x)=y(1-e^{-\beta x})^2e^{-\alpha x}, \end{aligned}\right. \tag{22}$$

where y and β are parameters to be determined by

$$\partial N/\partial y=0, \qquad \partial N/\partial\beta=0. \tag{23}$$

The integrations (18) and (19) can be carried through analytically and we used trial and error method to solve (23). First we calculated for $\xi=1$ and in some

* These trial wave functions correspond to choosing simply $\beta=\gamma$ in the equations (24) in Wu and Foley's paper, see reference 4. As can be seen from the results of Wu and Foley's, β is almost equal to γ and the quadrupole moment of the deuteron is almost entirely determined by y and is very insensitive to β and γ, so in order to simplify the calculation very much we choose specially $\beta=\gamma$, although the accuracy is reduced somewhat.

cases further examined the effect of changing ξ. Having obtained the values of y and β, we can readily calculate the quantities which can be compared with the empirical data mentioned at the beginning of this section. This is done below.

5. The effective range for the triplet and the singlet states:

Consider first the ^1S state of the continuum and let u_1 and u_2 be the solutions of (8) for two energies k_1^2 and k_2^2 that go asymptotically to the following functions

$$v_1 = \frac{\sin(k_1x+\delta_1)}{\sin\delta_1}, \qquad v_2 = \frac{\sin(k_2x+\delta_2)}{\sin\delta_2},$$

where δ_1 and δ_2 are the phase shifts. It can readily be shown from (8) and the differential equations for v_1 and v_2 that

$$k_2 \cot\delta_2 - k_1 \cot\delta_1 = (k_2^2 - k_1^2)\int_0^\infty (v_1v_2 - u_1u_2)\,dx \qquad (24)$$

and that the integral $\int_0^\infty (v_1v_2-u_1u_2)\,dx$ is rather insensitive to the value of the energies below a few Mev., and to a good approximation it can be regarded as a constant. This is used to define an effective range by

$$\frac{r_0}{2} \equiv \frac{1}{\varkappa}\int_0^\infty (v_1v_2 - u_1u_2)\,dx \cong \frac{1}{\varkappa}\int_0^\infty (v^2 - u^2)\,dx \qquad (25)$$

where u and v are the function for any energy below a few Mev. In order that any assumed nucleon potential $V(r)$ may reproduce the observed dependence of the phase δ on the energy k^2, it is necessary that the r_0 calculated from the solution u for the $V(r)$ be the same as the empirical value that is obtained from the empirical data cited before.

With the solution (13), we obtain

$$\varkappa r_0 = [3 - (3/4)\beta - (1/6)\beta^2] + k\cot\delta_0(4-3\beta), \qquad (26)$$

where β is the parameter in (13). Considering that k is of the order of 10^{-4}, one obtains from (11)

$$k\cot\delta_0 \cong k/\sin\delta_0 = 0.0556. \qquad (11')$$

If we substitute this value and $\beta=0.4955$ from (15) into (26), we obtain

$$r_0 = 3.2822\times 10^{-13}\text{cm} \qquad (27)$$

which is naturally the value for $m_\pi = 286 m_e$. Discussions about this is given later.

For the triplet states in the case of a purely central potential, the scattering at energies below a few Mev. can again be described by means of a similarly defined effective range r_0. In our present case with tensor forces, the situation is more complicated. It has been shown by Schwinger that to a good approximation, the following relation holds[4)]

$$k \cot \delta_0 = -u + (u^2+k^2)\frac{1}{\varkappa}\int_0^\infty [v_\alpha^2 - (u^2+w^2)]dx, \qquad (28)$$

where u and w are the solutions of (16) but normalized in such a way that u^2 goes asymptotically to $v_\alpha^2 = e^{-2\alpha x}$. From (28), it is clear that effective range r_0 is given by the following formula:

$$(1/2)\varkappa r_0 \equiv \int_0^\infty [e^{-2\alpha x} - (u^2+w^2)]dx, \qquad (29)$$

where we can use the expressions (22) for the the above u and w. Above integration is very elementary and one obtains r_0 as soon as β and y are given. The results are given later.

6. Quadrupole moment of the deuteron is given by

$$Q = (\sqrt{2}/10\varkappa^2)\int_0^\infty x^2(u-(1/2\sqrt{2})w)w\,dx,$$

where u and w are the solutions of (16) normalized according to

$$\int_0^\infty (u^2+w^2)\,dx = 1.$$

In our calculations, the variational wave functions (22) have been employed. The results are shown in the next section.

7. D-state probability:

D-state probability P_D is given by

$$P_D = \int_0^\infty w^2 dx$$

where wave functions are so normalized that

$$\int_0^\infty (u^2+w^2)\,dx = 1.$$

8. Slow proton-proton scattering and high energy neutron-proton scattering were not considered in this paper because there is no reason why neutron-proton force must be identical with that between protons, not only from theoretical point of view but also from experimental side, and in high energy phenomena some new complicated situations may occur which did not appear in low energy regions and so it seems worthwhile to limit the consideration to low energy neutron-proton interactions and exclude the discussions in high energy regions in order to examine the validity of Bethe's neutral vector meson theory and to seek what kind of potential is the most appropriate one for low energy neutron-proton interactions.

§ 3. Results and discussions

1. Vector case:

For $N=\xi=1$, the values of a and b which satisfy the equations (16) are

$$a=-23.02, \quad b=-10.73. \tag{30}$$

Above values mean that any positive values of a and b cannot bind deuteron and this can be anticipated from the fact that central repulsive force is larger than the non-central attractive force for $N=\xi=1$. In order to reduce the harmful repulsive central force, we then took $a=0$ after the Bethe's proposal, remaining N to be determined from deuteron binding. The results are given in Table I. As can be seen from the table, we must choose N very large in order to bind deuteron and this results in making tensor force too strong. This feature is not changed by letting ξ large, rather this has a serious disadvantage to make r_0 too small. From these calculations it is decisively concluded that the potential derived from neutral vector meson field is entirely unsatisfactory for deuteron system owing to its repulsive central force. In Bethe's calculations,[6] on the contrary, this repulsive force was effective to bind deuteron well. However, to the author's opinion, this success is accidental owing to the use of cut-off procedure that makes tensor force far stronger near the origin than in the case of simple Yukawa potential.

Table I

ξ	β	y	N	$Q(10^{-27}\text{cm}^2)$	$P_D(\%)$	$r_0(10^{-13}\text{cm})$
Neutral vector case						
1	2.00	0.458	13.66	6.349	15.00	0.961
3	1.00	0.375	35.60	4.318	11.73	0.016
Symmetrical pseudoscalar case						
1	1.80	0.274	1.08	4.678	5.84	1.416
2	2.83	0.260	2.15	3.944	5.63	0.940
Exp.				2.73	3.	1.56

2. Symmetrical pseudoscalar case:

Among the many potentials derived from current meson theory, the only one which has right sign of tensor force and has sufficiently large attractive central force is that of symmetrical pseudoscalar meson field, in agreement with the present most reliable conclusion concerning the type of π-meson drawn from various other considerations. In this case central force and non-central force are both attractive and have almost equal magnitude. The results are shown in Table I. In this case, too large values of Q and P_D and too small value of r_0 are considerably improved, but Q and P_D are still somewhat too large. Again, by changing ξ, main characteristics are not altered.

3. In order to see whether it is a general feature of adopting Yukawa type potential for tensor force that too large values of Q and P_D are obtained, or we can attain satisfactory agreement by making central force strong to some extent

but not abnormally strong, we introduced another factor k for central force in equation (4) and examined the effect of changing k from 1 to 1.5. In these calculations we choose $\xi=1$ for simplicity. The results are given in Table II.

Table II

k	β	v	N	$Q(10^{-27}\text{cm}^2)$	$P_D(\%)$	$r_0(10^{-13}\text{cm})$
1	1.80	0.274	1.08	4.678	5.84	1.416
1.3	1.70	0.185	0.67	3.436	2.72	1.575
1.39	1.660	0.142	0.50	2.733	1.62	1.634
1.4	1.656	0.136	0.47	2.629	1.48	1.640
1.5	1.60	0.04	0.13	0.820	0.13	1.716
Exp.				2.73	3.	1.56

As is readily seen from Table II, the values of Q and P_D are very sensitive to the change of k, namely the strength of central force. In order to fit Q value with experimental one, one has only to multiply central force by a factor of 1.39. In this case $N=0.50$, so relative weight of central force to tensor force is $1.39/0.50=2.8$. It can be said that this relative weight has the most decisive effect on the expression of nuclear potential and such a potential that explains deuteron problem well consists of large central force and relatively small tensor force; the main contribution to deuteron binding comes from strong attractive central part of force. Also from these conclusions neutral vector field which has repulsive central force is entirely excluded. The value of r_0 is roughly in agreement with empirical one, but P_D is too small compared with the empirical value 3%. This defect can be improved by considering the effect of changing ξ somewhat larger than 1, at least qualitatively. But quantitative estimation is not yet carried out.

The quantitative statements are very sensitive to the value of π-meson mass. If we stand on purely phenomenological ground and take $m_\pi=326.2m_e$,[16] the value which is demanded in order to explain the slow proton-proton scattering data by virtue of Yukawa type attractive potential, the numerical results will be modified to some extent and especially the relative weight of central to tensor force might possibly be near 1. However, the conclusion that sufficiently strong central force, at least the same order of magnitude as the tensor force, is necessary for deuteron system, will never be changed. To examine the effect of changing mass of π-meson to $326.2m_e$ is very interesting and are now in progress.

4. The theoretical value of singlet effective range r_0 is 3.28×10^{-13}cm, as is given by (27), for all the previously examined cases. At present any decisive statement about this quantity cannot be said from empirical data, so nothing can be said about the value 3.28×10^{-13}cm. But this is considerably larger than 2.71×10^{-13}cm, effective range for ^1S proton-proton interaction. Concerning this

point it is interesting to note that according to rough estimation $r_0=2.88\times10^{-13}$cm is obtained if we take $m_\pi=326.2m_e$. This shows charge independence of nuclear force, which is a well established fact for low energy nucleon interactions, indicating that $r_0=3.28\times10^{-13}$cm is reasonable value for $m_\pi=286m_e$.

§ 4. Conclusions

If the convensional cut-off procedure is not reasonable but an alternative approximation method which employs Yukawa type potential for irregular tensor force is more reasonable from meson theoretical point of view, the following conclusions can be drawn from the present calculations :

1. *Neutral vector meson field* which gives rise to *repulsive* central force in addition to tensor force of right sign, must be excluded entirely *owing to its strong repulsive central force.*

2. Among many possibilities of the types and combinations of meson fields, *symmetrical pseudoscalar meson field* is *the only one* which is satisfactory concerning deuteron problem and low energy neutron-proton interactions.

3. Considerably large attractive central force, at least the same order of magnitude as the tensor force of right sign, is decisively necessary and *the relative weight* of central force to tensor force has very much influence upon the value of quadrupole moment, D-state probability and triplet effective range. The best agreement was obtained with 2.8 for the above relative weight. But because this value is sensitive to the choice of π-meson mass, nothing decisive can be said about this value numerically but *the relative weight is certainly greater than* 1.

References

1) W. Rarita and J. Schwinger, Phys. Rev. **59** (1941), 436.
2) J. Schwinger, Phys. Rev. **61** (1942), 287.
3) J. M. Jauch and N. Hu, Phys. Rev. **65** (1944), 289.
4) Ta-You Wu and H. M. Foley, Phys. Rev. **75** (1949), 1681.
5) M. Sugawara, Prog. Theor. Phys. **3** (1948), 442.
6) H. A. Bethe, Phys. Rev. **57** (1940), 260, 390.
7) J. Schwinger, Phys. Rev. **72** (1947), 742.
8) L. Van Hove, Phys. Rev. **75** (1949), 1519; G. Araki, Phys. Rev. **75** (1949), 1101, 1262; Y. Nambu, Prog. Theor. Phys. **3** (1948), 444.
9) H. A. Bethe, Phys. Rev. **76** (1949), 38.
10) D. Melkonian, Phys. Rev. **76** (1949), 1744.
11) J. M. Blatt and J. D. Jackson, Phys. Rev. **76** (1949), 18.
12) Kellogg, Rabi, Ramsey and Zacharias, Phys. Rev. **57** (1940), 677.
13) H. A. Bethe, *Elementary Nuclear Theory* (1947).
14) Sutton, Hall, Anderson, Bridge, De Wire, Lavatelli, Long, Snyder and Williams, Phys. Rev. **72** (1947), 1147.
15) Hulthén, K. fysiogr. Sällsk. Lund. Förh. **14**, No. 21 (1944); **15**, No. 22 (1945).
16) L. E. Hoisington, S. S. Share and G. Breit, Phys. Rev. **56** (1939), 884.

Progress of Theoretical Physics, Vol. 5 No. 6 November~December, 1950

On the Negative π-Meson Capture.

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(Received May 24, 1950)

§ 1. Introduction

We discussed the meson production in the previous papers[1], but a definite conclusion on the type of mesons was not obtained. In this paper we will treat the annihilation of a meson, which will give us another information on the property of mesons.

When a positive π-meson is stopped in the matter, it is repelled from a nucleus by the Coulomb force and decays into a μ-meson. Therefore we shall be concerned here only with the negative π-meson.

After all the kinetic energy is dissipated by the ionization loss, the π^--meson is bound by the Coulomb field of a nucleus (cf. detailed discussions of Fermi and Teller[2]). Then it is absorbed by a proton in a nucleus before decaying into a μ^--meson, since they strongly interact with each other. In this case the following varieties of modes are possible:

(A) Processes which *can* occur even in a hydrogen atom,

$$\pi^- + p \rightarrow n + \gamma, \qquad (\gamma)$$

$$\pi^- + p \rightarrow n + \pi^0. \qquad (\pi)$$

(B) Processes which *can not* occur in a hydrogen atom,

$$\pi^- + p \rightarrow n, \qquad (n)$$

$$\pi^- + p + n \rightarrow n + n, \qquad (nn)$$

$$\pi^- + p + p \rightarrow n + p, \qquad (np)$$

$$\pi^- + p + n + n \rightarrow n + n + n, \qquad (nnn)$$

etc.,

where we denote a proton and a neutron by p and n. π^0 means a neutral meson capable to give rise to process (π)

The radiative capture of a meson, (γ), is the inverse process of meson production by γ-rays. The main difference between them is the magnitude of energy of a meson concerned. Thus the contribution of the meson current to the matrix element will be quite different for the capture and production of meson. The

γ-ray emitted in this process has energy of about μc^2, where μ means the mass of π^--meson.

The process (π) occurs only if the mass μ_0 of neutral meson is smaller than of charged meson. From the experiments at Berkeley, we can suppose that $\mu_0 \approx \mu$ and the neutral meson π^0 instantaneously decays into two γ-rays[3]. If this process (π) occurs, we shall observe two γ-rays with energy $\sim \mu c^2/2$.

The competition of (γ) and (π) is of special interest, for it has a close relation with the recent experiment of Panofsky[4]. Detailed discussion will be done later.

The process (n) is an analogy of the photo-electric effect. A bound proton in the nucleus can absorb a π^--meson, and starts with the energy $\sim \mu c^2$, changing into a neutron ("bound proton model"). As is seen, the Fourier component $\Phi(p)$ with $p \sim \sqrt{2M\mu}$ of the wave function $\Psi(r)$ of a bound proton is responsible to this process. However, this component $\Phi(p)$ will be considerably small, since the mean binding energy of a nucleus is much smaller than the rest energy μc^2 of a π^--meson. This means that the nuclear binding of nucleons may be regarded as weak in the annihilation process. Thus it seems necessary to examine an opposite limiting case of the free particle model for a nucleus. Then the momentum balance requires several recoil nucleons and the processes (nn), (np), (nnn) etc. occur. The lowest order process of them are (nn) and (np), which accompany only one recoil nucleon. In these cases, two nucleons in the final state have, on the average, energy $\sim \mu c^2/2$ and start in opposite directions. These processes are the inverse of meson production by nucleon-nucleon collision.

The competition of (n) and (nn)+(np) is an interesting problem, for detailed experimental studies on meson stars were carried out by various authors[5]. But it is regrettable that we can hardly obtain the reliable results for their probabilities of many ambiguities concerning the choice of $\Phi(p)$ (i.e. a model for a nucleus) and the nuclear force or the mesonic interaction.

Furthermore, the competition of (A) and (B) is also important. It has been established experimentally that most of negative mesons stopped in emulsion produce stars which will be caused by (B). Our calculation shows that the sum of the probabilities for (n), (nn), and (np) is considerably larger than that for (γ). Therefore, if the probability of process (π) is of the order of that of (γ), our results will not contradict with observed data.

§ 2. The process (γ) radiative capture

As was stated above, this process is possible even in a hydrogen atom. Therefore we calculate first for the hydrogen case, and obtain its probability W_γ as follows:

$$W_\gamma = \pi \frac{e^2}{\hbar c} \frac{f^2}{\hbar c} [illegible] \frac{\hbar}{M+\mu} \Sigma [illegible] \qquad (2.1)$$

where μ and M are the mass of π-meson and nucleon, and ψ_0 is the value of the wave function of a bound meson at the origin.

Table I. The Values of $Sp(\gamma)$.

Type of meson (Type of coupling)	Interaction	$Sp(\gamma)$
S(s)	$\sqrt{2}\sqrt{4\pi}\, f(\Psi^+\tau_-\Psi)\varphi+\text{c.c.}$	$4+o(\mu^2/M^2)$
S(v)	$\sqrt{2}\sqrt{4\pi}\, g(\Psi^+\tau_-\gamma_\mu\Psi)\partial_\mu\varphi+\text{c.c.}$	0
Ps(ps)	$i\sqrt{2}\sqrt{4\pi}\, f(\Psi^+\tau_-\gamma_5\Psi)\varphi+\text{c.c.}$	$4+o(\mu^2/M^2)$
Ps(pv)	$i\sqrt{2}\sqrt{4\pi}\, g(\Psi^+\tau_-\gamma_5\gamma_\mu\Psi)\partial_\mu\varphi+\text{c.c.}$	$16\ M^2/\mu^2$

s, v, ps and pv are the abbreviation of scalar, vector, pseudo-scalar and pseudo-vector. In the first column of the Table I, we denote the type of the meson field before the bracket, and the type of the coupling in it. For example, Ps(pv) means the pseudo-scalar meson with the pseudo-vector coupling. Of course, the factor $f^2/\hbar c$ in (2.1) should be replaced by $g^2/\hbar c$ in cases of v and pv coupling.

In the second column, Ψ means the nucleon wave function and φ describes the meson field. Other notations are the same as used by Fukuda and Takeda, ref. 1). $f^2/\hbar c$ or $g^2/\hbar c$ is a dimensionless coupling constant to be compared with $e^2/\hbar c=1/137$.

Finally, we note that for a bound meson $\varphi=\frac{\hbar}{\sqrt{2\mu}}\psi$, where ψ is normalized as $\int|\psi|^2 d\boldsymbol{r}=1$.

For example, if we take the K wave function for a bound meson*), $|\psi_0|^2$ becomes simply,

$$|\psi_0|^2=\frac{1}{\pi}\left(\frac{\mu c}{\hbar}\right)^3\left(\frac{e^2}{\hbar c}\right)^3.$$

Then the numerical value of $W_\gamma^{(k)}$ can be given as follows (cf. Table I) : for the meson of the type S(s) or Ps(s),

$$W_\gamma^{(k)}=\frac{f^2}{\hbar c}\times 4.8\times 10^{13}\ \text{sec}^{-1},$$

and for the meson of the type Ps(pv),

$$W_\gamma^{(k)}=\frac{g^2}{\hbar c}\times 0.83\times 10^{16}\ \text{sec}^{-1}. \tag{2.2}$$

For the meson capture by an atom other than a hydrogen, we can obtain an approximate probability of (γ) per one proton in a nucleus, by suitably modifying (2.1) in the density of proton. The results are ; for S(s) or Ps(ps), (Z=*atomic number*)

*) For definiteness, we refer to the K wave function, though a meson will probably be captured from an outer orbit.

$$\left.\begin{aligned} &\frac{1}{Z}W_\gamma=\frac{f^2}{\hbar c}|\psi_0|^2 r_0^3\times 4.24\times 10^{20}\ \text{sec}^{-1}, \\ &\text{and for Ps(pv),} \\ &\frac{1}{Z}W_\gamma=\frac{g^2}{\hbar c}|\psi_0|^2 r_0^3\times 7.22\times 10^{22}\ \text{sec}^{-1}, \end{aligned}\right\} \tag{2.3}$$

where $r_0=1.37\times 10^{-13}$ cm is connected by the relation $R=r_0A^{1/3}$ with the radius R of the nucleus with mass number $A=2Z$ and

$$|\psi_0|^2=\frac{1}{\frac{4\pi}{3}R^3}\int_{|r|\leq R}|\psi(\boldsymbol{r})|^2 d\boldsymbol{r}.$$

The above approximation corresponds to the neglection of the nuclear binding for a proton in a nucleus. The effect of nuclear binding will not change the absolute probability more than a factor 2 or 3. This can be supposed by the following estimation. Assuming the Gaussian type for a wave function of a bound proton, the probability is given by (cf. eq. (5.1 G) taking $a=r_0$)

$$Z\frac{f^2}{\hbar c}|\psi_0|^2 r_0^3\times 8.24\times 10^{20}\ \text{sec}^{-1}\quad \text{for}\quad \text{S(s)}.$$

The probability for π-μ decay is experimentally determined as $\sim 10^{+8}$ sec^{-1}. We can show that this is smaller than the probability of (γ) (cf. Fermi and Teller, ref. 2)).

This process was discussed in greater details by Bruno[7].

§ 3. The process (π)

The process (π) can occur even in a hydrogen atom, competing with the process (γ). Its probability for hydrogen case is calculated as,

$$W_\pi=\pi|\psi_0|^2 c\frac{\hbar}{Mc}\,\frac{\hbar}{\mu c}\,\frac{f^2}{\hbar c}\,\frac{f_0^2}{\hbar c}\,\frac{\sqrt{\mu^2-\mu_0^2}}{M+\mu}\,Sp(\pi), \tag{3.1}$$

where f_0 and μ_0 are the coupling constant and the mass of a neutral meson. The value of $Sp(\pi)$ is different for different types of meson assumed, as is shown in Table II.

We can apply the above results for an atom other than hydrogen, with analogous modification as discussed in § 2. These processes, (γ) and (π), were also discussed by Nagoya group[6].

§ 5. Discussion I···Capture by hydrogen atom

In the preliminary experiment, Panofsky observed the γ-rays with energies

Table II. The Value of $Sp(\pi)$.

Type of mesons		$Sp(\pi)$	
π^-	π°	Neutral type interaction	Symmetrical type interaction
S(s)	S(s)	$2(\mu^2-\mu_0^2)^2/\mu^2 M^2$	
S(s)	S(s)	0	$32\ M^2/\mu^2$
S(s)	S(v)	0	
S(v)	S(v)	0	
Ps(ps)	Ps(ps)	8	$2\mu^2/M^2$
Ps(pv)	Ps(ps)	$2\mu^2/M^2$	‖
Ps(ps)	Ps(pv)	‖	8
Ps(pv)	Ps(pv)	8	$2[2(\mu_0^2-\mu^2)M+\mu_0^2\mu]^2/M^2\mu^4$
S(s)	Ps(ps)	$(2\mu^2-\mu_0^2)/M^2$	$8(\mu^2-\mu_0^2)/\mu^2$
S(v)	Ps(ps)	0	‖
S(s)	Ps(pv)	$2(\mu^2-\mu_0^2)/M^2$	
S(v)	Ps(pv)	0	$32\ (\mu^2-\mu_0^2)M^2/\mu^4$
Ps(ps)	S(s)	$(\mu^2-\mu_0^2)\mu^2/2M^4$	$2(\mu^2-\mu_0^2)/M^2$
Ps(pv)	S(v)	$8(\mu^2-\mu_0^2)/\mu^2$	
Ps(ps)	S(v)	0	$8(\mu^2-\mu_0^2)/\mu^2$
Ps(pv)	S(v)	0	

‖ denotes the set between each of which the equivalence relation holds. We assume two types of τ-interactions between nucleons and neutral mesons;

(1) neutral type, e.g., $\sqrt{4\pi}\, f_0\ (\Psi^+\tau_3\Psi)\ \varphi_0$ for S(s),

(2) symmetrical type, e.g., $\sqrt{4\pi}\, f_0\ (\Psi^+\Psi)\varphi_0$ for S(s).

$\sim$65 MeV and$\sim$130 MeV from π^--mesons stopped in the hydrogen gas[4]. As was pointed by him, this may be interpreted due to the processes (γ) and (π) followed by the two γ-decay of π^0. The direct emission probability of two γ-rays is found by Ogawa and Yamada[6] to be as small as one fiftieth of that of (γ). Therefore we may conclude that i) the processes (γ) and (π) occur with comparable frequencies in hydrogen gas and ii) their probabilities are larger than that of π-μ decay. From the calculations in §2 and §3, we obtain their ratio W_π/W_γ as, (cf. Table III),

$$\frac{W_\pi}{W_\gamma}=\frac{f_0^2}{e^2}\,\frac{\sqrt{\mu^2-\mu_0^2}}{\mu}\,\frac{Sp(\pi)}{Sp(\gamma)}. \tag{4.1}$$

As we can suppose $\mu_0\approx\mu$ (cf. ref. 3)), their mass difference is put as $\Delta\mu=\mu-\mu_0$ (more precisely, $\Delta\mu=\mu-\mu_0-$ (neutron mass) + (proton mass) − (electro-static binding energy of a bound π^--meson)), and may be regarded as $\Delta\mu\ll\mu$.

Table III. Values of $Sp(\gamma)/Sp(\pi)$.

Type of mesons		$Sp(\gamma)/Sp(\pi)$	
π^-	π^0	Neutral type interaction	Symmetrical type interaction
S(s)	S(s)	$M^2/2(\Delta\mu)^2$	$\mu^2/8M^2$
S(v)	S(s)	—	—
S(s)	S(v)	—	$\mu^2/8M^2$
S(v)	S(v)	—	—
Ps(ps)	Ps(ps)	1/2	$2\ M^2/\mu^2$
Ps(pv)	Ps(ps)	$8\ M^4/\mu^4$	$2\ M^4/\mu^4$
Ps(ps)	Ps(pv)	$2\ M^2/\mu^2$	1/2
Ps(pv)	Ps(pv)	$2\ M^2/\mu^2$	$8M^4/[\mu_0^2-4\Delta\mu M]^2$
S(s)	Ps(ps)	$M^2/\mu\Delta\mu$	$\mu/4\Delta\mu$
S(v)	Ps(ps)	—	—
S(s)	Ps(pv)	$M^2/\mu\Delta\mu$	$\mu^2/16\ M^2\Delta\mu$
S(v)	Ps(pv)	—	—
Ps(ps)	S(s)	$4\ M^4/\mu^3\Delta\mu$	$M^2/\mu\Delta\mu$
Ps(pv)	S(s)	$M^2/\mu\Delta\mu$	$M^2/\mu\Delta\mu$
Ps(ps)	S(v)	—	$\mu/4\Delta\mu$
Ps(pv)	S(v)	—	$M^2/\mu\Delta\mu$

In the relation (4.1), we have three undetermined quantities $\Delta\mu, f_0$ (g_0) and W_π/W_γ. We may consider from production experiment[3] that f_0 is the same order of the magnitude as f, though their absolute values can not be determined unambiguously, as seen in our accompanied work[1]. On the other hand, $\Delta\mu$ will be fixed by increasing accuracy of experiments. In fact, the width ΔE of the energy spectrum of γ-rays around $\mu c^2/2$ will be given by,

$$\Delta E \sim (2\Delta\mu/\mu)^{\frac{1}{2}}\mu c^2 \sim \text{a few Mev.} \tag{4.2}$$

Furthermore, W_π/W_γ may be nearly equal to 1.

Then the coupling constant f_0 will be estimated from the equation,

$$\frac{f_0^2}{e^2}=\left(\frac{\mu}{2.\Delta\mu}\right)^{\frac{1}{2}}\frac{Sp(\gamma)}{Sp(\pi)}\cdot\frac{W_\pi}{W_\gamma}. \tag{4.3}$$

Since the ratio $Sp(\gamma)/Sp(\pi)$ is elementarily evaluated as is shown in Table III, we shall have an information about the type and the magnitude of coupling constant of mesonic interaction. We can just reject the case where both π^--and π^0-mesons are scalar, because of the following reasons:

(a) Charged meson S(s) and neutral meson S(s) of neutral type. In this case (4.3) results in

$$\frac{f_0^2}{e^2} \approx \frac{1}{\sqrt{8}}\left(\frac{\mu}{\Delta\mu}\right)^{\frac{5}{2}}\left(\frac{M}{\mu}\right)^2 = 16\left(\frac{\mu}{\Delta\mu}\right)^{\frac{5}{2}}$$

for $W_\pi \approx W_\gamma$. The coupling constant $f_0^2/\hbar c$ can be reduced to $\gtrsim 1$, only if $\Delta\mu/\mu \gtrsim 1/2.3$. This mass difference seems to be too large to be reconciled with the magnitude of ΔE and other experimental facts concerning neutral meson[3)].

(b) Charged meson S(s) and neutral meson S(s) or S(v) of symmetrical type. Here we find

$$\frac{f_0^2}{e^2} \approx \frac{1}{\sqrt{2}}\left(\frac{\mu}{\Delta\mu}\right)^{\frac{1}{2}}\frac{\mu^2}{8M^2} \sim 2\times 10^{-3}\left(\frac{\mu}{\Delta\mu}\right)^{\frac{1}{2}}$$

In order to get $f_0^2/\hbar c$ of the order of unity, $\Delta\mu/\mu$ must be as small as $\sim 10^{-10}$. Such a small mass difference is very unlikely, although the measurement of the width ΔE may not be so accurate as to decide whether the mass difference is really so small as above or not.

(c) In the other cases, either W_π or W_γ vanishes and two groups of γ-rays from π^--capture by hydrogen should not be observed.

By the similar arguments we may reject the cases, when either π^- or π^0 is scalar.

Thus we are compelled to adopt the pseudo-scalar meson theory both for the charged and neutral mesons. Furthermore, if we could know the precise value of μ_0 and W_γ/W_π, we could determine the coupling constant f_0. For example, if we tentatively take

$$\frac{\Delta\mu}{\mu} = \frac{3}{100}(\Delta E \approx 30\,\text{Mev}), \quad \frac{W_\pi}{W_\gamma} = 1,$$

we get

$$\frac{f_0^2}{\hbar c} = 4\,\frac{1}{137}\,\frac{Sp(\gamma)}{Sp(\pi)}.$$

The results obtained from this expression are tabulated in Table IV.

Table IV. The Values of Coupling Constants.

Type of meson		Coupling constant*)	
π^-	π^0	Neutral type interaction	Symmetrical type interaction
Ps(ps)	Ps(ps)	0.016	2.5
Ps(pv)	Ps(ps)	440	2.5
Ps(ps)	Ps(pv)	2.5	0.015
Ps(pv)	Ps(pv)	2.5	>440

*) The value of $f_0^2/\hbar c$ for ps-coupling, and $g^2/\hbar c$ for pv-coupling. In this table, we tentatively take $\Delta\mu/\mu = 3/100$ ($\Delta\mu = 4.3$ MeV/c^2) and $W_\pi/W_\gamma = 1$.

§ 5. The process (n) ⋯ Analogue of the photo-electric effect

We will here estimate the order of magnitude of the probability W_1 for the process (n), which has been discussed by many authors[7]. We take the following three types for the wave function of a bound proton;
Gaussian type

$$\Psi_p = \frac{1}{\pi^{3/4} a^{3/2}} e^{-\frac{1}{2}(r/a)^2}, \tag{5.1G}$$

exponential type

$$\Psi_p(r) = \frac{1}{\pi^{1/2} a^{3/2}} \; e^{-r/a} \tag{5.1E}$$

and Yukawa type

$$\Psi_p(r) = \frac{1}{1^{1/2} \pi^{1/2} a^{1/2}} \frac{e^{-r/a}}{r}. \tag{5.2Y}$$

The use of these wave function compells us to restrict our calculation within the non-relativistic or Pauli approximation.

I) **Non-relativistic case.** As for the mesonic interaction, we assume the following two types:

$$\sqrt{2}\,\sqrt{4\pi}\, f\, (\Psi^* \tau_- \Psi_p)\, \frac{\hbar}{\sqrt{2\mu}}\, \psi \qquad \text{S(s, N.R.)}$$

and

$$\frac{\hbar}{\mu c} \sqrt{2}\,\sqrt{4\pi}\, g\, (\Psi^* \tau_- \sigma \Psi_p)\, \frac{\hbar}{\sqrt{2\mu}}\, \Gamma\psi, \qquad \text{Ps(pv, N.R.)}$$

where ψ is the wave function of a bound meson, normalized as $\int |\psi(\boldsymbol{r})|^2 d\boldsymbol{r} = 1$.

(*Note added in proof:* We have overlooked here the coupling containing time-derivative, which leads to the same results as obtained from Ps (pv, Pauli) (see next page)).

We neglect the small binding energy of a meson caused by the electrostatic potential of the nucleus. Then the probability W_1 per unit time per one nucleus for the process (n) is given by,

$$W_1 = 2^{\frac{11}{2}} \pi \cdot Z \cdot (\;) \cdot \frac{Mc^2}{\hbar} \left(\frac{M}{\mu}\right)^{\frac{1}{2}} \cdot \times$$

$$\times \begin{cases} \pi^{\frac{1}{2}} e^{-L} & \text{for Gaussian type,} \\ \dfrac{2}{(L+1)^4} & \text{for exponential type,} \\ \dfrac{1}{(L+1)^4} & \text{for Yukawa type,} \end{cases}$$

where

Z=atomic number of the nucleus,

$$L=\frac{a^2}{r_0^2}\frac{\mu c^2}{K}, \qquad K=\frac{\hbar^2}{2Mr_0^2}=11.1 \text{ Mev},$$

and ( ) means,

$$(\quad)=\frac{f^2}{\hbar c}|\psi_0|^2 a^3 \qquad \text{for S(s, N.R.)},$$

$$(\quad)=\frac{g^2}{\hbar c}\left|\frac{\hbar}{\mu c}\Delta\psi_0\right|^2 a^3 \qquad \text{for Ps(pv, N.R.)},$$

$$|\psi_0|^2=\frac{1}{\frac{4\pi}{3}R^3}\int_{|r|\leq R}|\psi(\boldsymbol{r})|^2 d\boldsymbol{r}.$$

If we take, for example, $a=r_0$, we get;

$$W_1=Z\cdot(\quad)\times\begin{cases}2.03\times10^{21}\ \text{sec}^{-1} & \text{for Gaussian type,}\\ 2.70\times10^{22}\ \text{sec}^{-1} & \text{for exponential type,}\\ 2.63\times10^{24}\ \text{sec}^{-1} & \text{for Yukawa type,}\end{cases}$$

and the cross-sections are given by,

$$\sigma_1=Z\cdot\frac{c}{v}\cdot(\quad)\times\begin{cases}5.22\times10^{-28}\ \text{cm}^2 & \text{for Gaussian type,}\\ 6.94\times10^{-27}\ \text{cm}^2 & \text{fer exponential type,}\\ 6.78\times10^{-25}\ \text{cm}^2 & \text{for Yukawa type,}\end{cases}$$

where v means the incident velocity of the negative meson.

We remark that the Yukawa wave function with $a\approx r_0$ is ascertained by Chew and Goldberger[8], while the Gaussian wave function were often used to calculate the total binding energy of a nucleus. Thus we may consider the true value lies between the cases of Gaussian and Yukawa types, say,

$$W_1=Z\cdot(\quad)\times10^{22}\sim10^{23}\ \text{sec}^{-1}.$$

II) **Pauli approximation** (for details, see Bruno, ref. 7)). For Ps(ps), the Pauli approximation gives the probability W_1 which is smaller than (6.2) for $S(s)$ by the factor $\mu/2M$:

$$W_1=Z\cdot\frac{f^2}{\hbar c}|\psi_0|^2 r_0^3\times\begin{cases}1.54\times10^{20}\ \text{sec}^{-1} & \text{for Gaussian type,}\\ 2.04\times10^{21}\ \text{sec}^{-1} & \text{for exponential type,}\\ 2.00\times10^{23}\ \text{sec}^{-1} & \text{for Yukawa type.}\end{cases}$$

Ps(ps, Pauli)

§ 6. The processes (nn) and (np)

These absorption processes were previously pointed out when we discussed the nuclear disintegrations caused by them[9]. The total momentum is conserved by one recoil nucleon and this recoil momentum is transferred by the nuclear force.

We calculate the probability for these processes, in following two ways. i) We assume the phenomenological nuclear force and use the meson theory only for meson annihilation, as was done by Foldy and Marshak in their treatment on meson production[10]. ii) We calculate the whole process meson-theoretically. The Fermi gas model is assumed for a nucleus.

i) Using the phenomenological nuclear force. We adopt the following Yukawa potential as the nuclear force. (Cf. ref 10))

$$V(r) = JOw(\varkappa r), \tag{6.1}$$

where

$$w(x) = \frac{e^{-x}}{x}$$

and

$$O = -\frac{1}{4}[(1+P_\sigma)(1-P_\tau) + (1-P_\sigma)(1+P_\tau)q].$$

J is the depth of the potential in ^3S-state, and q is the ratio of the potential depth in ^1S to that in ^3S. To fit the neutron-proton scattering and the binding energy of deuteron, they are determined as,[10]

$$\begin{cases} J = 67.3 \text{ Mev}, \\ \hbar c\varkappa = 167 \text{ Mev}, \\ q = 0.693, \quad q^2 = 0.480. \end{cases} \tag{6.1 a}$$

The absorption probability W_2 can be separated into two parts; W_{nn} for the process (nn) and W_{np} for the process (np). Calculations are carried out in the case Ps (pv). The results are as follows; If we adopt the non-relativistic interaction (cf. § 5), we obtain, for Ps(pv; N.R., N.F.),

$$W_2 = W_{nn} + W_{np} = \frac{4}{3} W_{nn} = 4 W_{np} \tag{6.2}$$

$$= \frac{g^2}{\hbar c} \cdot \int_{|r| \leq R} \left| \frac{\hbar}{\mu c} \Delta\psi \right|^2 dr \cdot \frac{(1-q)^2}{4\pi} \left(\frac{J}{\mu c^2}\right)^2 \frac{M}{\mu} \frac{Mc^2}{\hbar} I.$$

I is the quadruple integral in momentum space, which appears from averaging over the initial momentum of nucleons through Fermi distribution and summing up over the final momentum outside the occupied sphere;

$$I=\iiiint\limits_{\substack{|\boldsymbol{P}_1^0|,\,|\boldsymbol{P}_2^0|\leqq p_F\\ |\boldsymbol{P}_1|,\,|\boldsymbol{P}_2|\leqq p_F}} \frac{d\boldsymbol{P}_1^0 d\boldsymbol{P}_2^0}{\left(\frac{4\pi}{3}p_F^3\right)^2}\frac{d\boldsymbol{P}_1 d\boldsymbol{P}_2}{M^2c}\,\delta(\boldsymbol{P}_1^0+\boldsymbol{P}_2^0-\boldsymbol{P}_1-\boldsymbol{P}_2).$$

$$\cdot\delta\left(\frac{\boldsymbol{P}_1^{02}}{2M}+\frac{\boldsymbol{P}_2^{02}}{2M}+\mu c^2-\frac{\boldsymbol{P}_1^2}{2M}-\frac{\boldsymbol{P}_2^2}{2M}\right).$$

$$\cdot[w_p(|\boldsymbol{P}_1^0-\boldsymbol{P}_1|)+w_p(|\boldsymbol{P}_1^0-\boldsymbol{P}_2|)]^2. \tag{6.2 a}$$

Momenta with and without suffix 0 are those of the initial and the final state, respectively. p_F means the Fermi momentum of nucleon gas and is $\sim$200 MeV/c. $w_p(q)$ is the Fourier component of Yukawa potential multiplied by the nucleon density, i.e.,

$$w_p(q)=\frac{1}{\frac{4\pi}{3}r_0^3}\cdot\frac{2\pi}{\varkappa^3}\frac{2}{1+(q/\hbar\varkappa)^2}. \tag{6.2 b}$$

Approximate estimation of the integral I is given in Appendix. We see that this result is practically equal to the one which is obtained neglecting the Fermi distribution and assuming the two nucleons are initially at rest. The numerical value is;

$$W_2=\frac{g^2}{\hbar c}\int\limits_{|\boldsymbol{r}|\leqq R}\left|\frac{\hbar}{\mu c}\Delta\psi\right|^2 d\boldsymbol{r}\times 1.59\times 10^{22}\ \mathrm{sec}^{-1}$$

$$=Z\cdot\frac{g^2}{\hbar c}\left|\frac{\hbar}{\mu c}\Delta\psi_0\right|^2\cdot r_0^3\times 1.33\times 10^{23}\ \mathrm{sec}^{-1},$$

for Ps(pv, N.R., N.F.),

where

$$\left|\frac{\hbar}{\mu c}\Delta\psi_0\right|^2=\frac{1}{\frac{4\pi}{3}R^3}\int\limits_{|\boldsymbol{r}|\leqq R}\left|\frac{\hbar}{\mu c}\Delta\psi(\boldsymbol{r})\right|^2 d\boldsymbol{r},\quad R=r_0A^{1/3},$$

provided $A=2Z$.

If we take the non-relativistic interaction for S(s), we find,

$$W_2=0,\qquad \text{for S(s, N.R., N.F.)}.$$

On the other hand, covariant calculation with covariant interaction gives the following result for Ps(ps),

$$W_2=Z\cdot\frac{f^2}{\hbar c}|\psi^0|^2 r_0^3\times 1.78\times 10^{21}\ \mathrm{sec}^{-1},$$

$$W_{\nu p}=0.47\ W_2,$$

where

$$|\psi_0|^2=\frac{1}{\frac{4\pi}{3}R^3}\int_{|\boldsymbol{r}|\leq R}|\psi(\boldsymbol{r})|^2 d\boldsymbol{r} \qquad \text{Ps(ps, N.F.)},$$

and thus the equivalence theorem gives for Ps(pv),

$$W_2=Z\cdot\frac{g^2}{\hbar c}|\psi_0|^2\ r_0^3\times 3.15\times 10^{23}\ \text{sec}^{-1} \qquad \text{Ps(pv, N.F.)}.$$

On account of strong interaction of π-meson with nucleons, a nucleus can absorb a π^--meson bound in an outer orbit, and the effect of binding of a π^--meson itself can be neglected. While the effect of non-static interaction should not be overlooked and the covariant calculation must be used for our purpose, because of rather large rest energy of a π^--meson.

ii) Using the meson theory throughout the whole calculation. Assuming the symmetrical theory, we get the following results from covariant calculation,

$$W_2'=2W_{nn}'=2W_{np}'$$

$$=Z\left(\frac{f^2}{\hbar c}\right)^3|\psi_0|^2 r_0^3\times 3.2\times 10^{22}\ \text{sec}^{-1} \qquad \text{S(s)},$$

$$W_2=\frac{14}{13}W_{nn}=14W_{np}$$

$$=Z\left(\frac{g^2}{\hbar c}\right)^3|\psi_0|^2 r_0^3\times 2.14\times 10^{27}\ \text{sec}^{-1} \qquad \text{Ps(pv)}$$

and

$$W_2'=2W_{nn}'=2W_{np}'$$

$$=Z\left(\frac{f^2}{\hbar c}\right)^3|\psi_0|^2 r_0^3\times 4.8\times 10^{23}\ \text{sec}^{-1} \qquad \text{Ps (ps)}.$$

§ 7. Discussion II…Capture by an atom other than hydrogen

It is well known from the experiments on artificially produced mesons[11], that nearly 73% of negative mesons stopped in emulsion produce stars with more than one prong. Furthermore, about half of the π^--mesons not accompanied by an observable star prong show clubs at their end of the track, which seem to be due to recoil nuclei. Thus, we can consider the processes (n), (nn) and (np) etc. occur at least in 86% of meson-capturing nuclei. From the properties of meson stars, it is hard to conclude which of the process (n) or $(nn)+(np)$ is the main process. The fast protons and the prong spectrum of meson-stars can be roughly explained in either case, by calculations on the nuclear penetration and the nuclear

evaporation process, although all of the observed data seem to favor the process $(nn)+(np)$[12].

We shall discuss the results of the above calculations in comparison with these experimental facts (cf. Table *V*).

Table V. Summary of §5 and §6.

Process (n)	Type of meson		
	S(s)	Ps(ps)	Ps(pv)
W_1/Z ( ) N, R, or Pauli	2.0×10^{21} (G) 2.7×10^{22} (E) 2.6×10^{24} (Y)	1.5×10^{20} (G) 2.0×10^{21} (E) 2.6×10^{23} (Y)	2.0×10^{21} (G) 2.7×10^{22} (E) 2.6×10^{24} (Y)
Process $(nn)+(np)$			
$W_2/Z\,\frac{g^2}{\hbar c}\left\lvert\frac{\hbar}{\mu c}\Delta\phi\right\rvert^2 r^3$ N.R., N.F.	0		1.33×10^{23}
$W_2/Z\,\frac{f^2}{\hbar c}\lvert\phi^0\rvert^2 r_0^3$ N.F.		1.78×10^{21}	3.15×10^{22}
$W_2/Z\left(\frac{f^2}{\hbar c}\right)^3\lvert\phi_0\rvert^2 r_0^3$ meson-theoretical	3.2×10^{22}	2.1×10^{27}	4.8×10^{23}

1) The meson of type S(s). The probability per one proton for (n) is (in exponential or Yukawa case, cf. § 4)

$$\frac{f^2}{\hbar c}|\phi_0|^2 r_0^3\times 2.7\times10^{22}\sim2.6\times10^{24}\ \mathrm{sec}^{-1}$$

and that for $(nn)+(np)$ is

$$\left(\frac{f^2}{\hbar c}\right)^3|\phi_0|^2 r_0^3\times3.2\times10^{23}\ \mathrm{sec}^{-1}.$$

These are comparable order of magnitude. While for (γ), as mentioned in § 2

$$\frac{f^2}{\hbar c}|\phi_0|^2 r_0^3\times4.2\times10^{20}\ \mathrm{sec}^{-1}.$$

This is much smaller than that of (n) or $(nn)+(np)$, and thus the meson of the type S(s) does not seem to contradict with the experiments on meson-star.

2) The meson of type Ps(ps). As well as in the meson production, the probability of $(nn)+(np)$ differs greatly whether we use the phenomenological nuclear potential or not. The use of the phenomenological potential may be quite wrong, since it gives unreasonably large coupling constant $f^2/\hbar c=270$ in order to fit the meson production (cf. ref. 1)). If we use the meson theory throughout the process, the probability of $(nn)+(np)$ is given by,

$$\frac{1}{Z}W_2=\left(\frac{f^2}{\hbar c}\right)^3|\psi_0|^2r_0^3\times 4.8\times 10^{23}\ \mathrm{sec}^{-1}.$$

The coupling constant $f^2/\hbar c$ is determined as ~ 2.0, from the meson production by protons (cf. ref. 1)). While the probability for (γ) is,

$$\frac{1}{Z}W_\gamma=\frac{f^2}{\hbar c}|\psi_0|^2r_0^3\times 4.2\times 10^{20}\ \mathrm{sec}^{-1}.$$

These two W's should not be directly compared with each other, because these two are inverse processes of the meson production by a nucleon and a γ-ray, and the value of coupling constant $f^2/\hbar c$ determined by them differ considerably with each other, i.e. $f^2/\hbar c=17$ for the production by γ-rays and $f^2/\hbar c=2.0$ for that by protons. This may be due to the neglect of higher order processes, especially the 4-th order processes in γ-ray case, or the wrong approximation of perturbation method accounting for the large magnitude of coupling constant. Therefore we might insert different value of the coupling constant for W_2 and W_γ and compare them,

$$\frac{W_2}{W_\gamma}=\frac{2^3\times 4.8\times 10^{23}}{17\times 4.2\times 10^{20}}\doteqdot 50.$$

Apart from such an ambiguity in the magnitude of $f^2/\hbar c$, the meson of type Ps(ps) seems to be able to explain meson-stars, too. The probability for (n) is of the order of

$$\frac{1}{Z}W_1=\frac{f^2}{\hbar c}|\psi_0|^2r_0^3\times 10^{20}\sim 10^{23}\ \mathrm{sec}^{-1}.$$

Thus we have $W_1\gtrsim W_2$, and $(nn)+(np)$ will be the main process of the π^--meson capture.

3) The meson of type Ps(pv). From the above calculations, we find,

$$W_\gamma<W_2$$

both for Ps(pv) and Ps(pv), S(s) and the agreement with observation is certified.

§ 8. Conclusion

We summarized in Table VI the result obtained from the meson production and the meson capture which were discussed here and in the preceding papers[1] And the criteria for various type of mesons are tabulated as follows:
As is seen from this table, S(s) seems to be certainly rejected, since there are further reasons unfavorable to this type, e.g. the nuclear force and the anomalous magnetic moment, etc. Thus the pseudo-scalar meson theory seems to be most favorable, as considered heretofore, and the use of phenomenological potential may not be allowed for mesonic phenomena. However, there seems to remain ambiguity in the pseudo-scalar coupling constant of pseudo-scalar meson,

Table VI. Summary of Results.

Meson production by :	Type of meson			Experimental
	S(s)	Ps(pv)	Ps(ps)	
X-rays	$e^2f^2\times 6\times 10^{-27}$ ($f^2=1.08$)	($g^2\sim 0.1$)	$e^2f^2\times 3.82\times 10^{-28}$ ($f^2=17$)	4.2×10^{-29}
Protons (meson-theoretical)	$f^6\times 5.3\times 10^{-27}$ ($f^2=0.35$)	$g^6\times 3.4\times 10^{-24}$ ($g^2=0.19$)	$f^6\times 2.8\times 10^{-29}$ ($f^2=2.0$)	
Protons (Phenomenological nuclear force)	$\sim f^2\times 10^{-28}$ ($f^2\sim 2$)	$g^2\times 1.5\times 10^{-28}$ ($g^2=1.5$)	$f^2\times 8.4\times 10^{-31}$ ($f^2=270$)	2.3×10^{-28}
Meson Capture :				
(γ)	$f^2\times\begin{Bmatrix}4.2\\8.2\end{Bmatrix}\times 10^{20}$	$g^2\times 7.5\times 10^{22}$	$f^2\times 4.2\times 10^{20}$	
$(nn)+(np)$ (meson-theoretical)		$g^6\times 2.1\times 10^{27}$ (13 : 1)	$f^6\times 4.8\times 10^{23}$ (1 : 1)	
$(nn)+(np)$ (phenomenological nuclear force)	$f^6\times 3.2\times 10^{23}$ (1 : 1)	$g^2\times 3.2\times 10^{23}$ ($\sim$1 : 1)	$f^2\times 1.8\times 10^{21}$ ($\sim$1 : 1)	

The figures concerning meson production are the total cross-section per one nucleon in cm², and those concerning meson capture give the capture probability per one proton in the nucleus for various processes in sec⁻¹. divided by $|\psi_0|^2 r_0^3$.

The figures in parenthesis mean the ratio W_{nn}/W_{np}. Furthermoie, f^2 (or g^2) and e^2 are the abbreviation of dimensionless constants $f^2/\hbar c$ (or $g^2/\hbar c$) and $e^2/\hbar c$.

Table VII. Criteria.

Criterion	Type of meson		
	S(s)	Ps(pv)	Ps(ps)
1) Is the angular distribution of mesons of mesons from X-rays nearly isotropic?	No.	Yes.	Yes.
2) Are the coupling constants determined from meson production by X-rays and protons equal?	Yes.	Yes.	No.*
3) In capture process by an atom other than hydrogen, has (γ) smaller probability than $(n)+(nn)+(np)$ + etc.?	Yes.	Yes.	Yes.
4) In hydrogen, have the process (γ) and (π) comparable probability?	No.**	Yes.	Yes.
5) Does it favor the two γ-decay of a neutral meson ?[14]	Yes.	Yes.	Yes.
6) Does it favor the three π-decay of a τ-meson?	No.	Yes.	Yes.

*) The 4-th order calculation may revise this discrepancy.[12]

**) It may be possible that the case, (see § 4),

$$\begin{cases}\text{charged meson: S(s),}\\ \text{neutral meson : S(s) or S(v) of symmetrical type.}\end{cases}$$

which may not be solved without the further progress of experiments or the more refinement of the present formalism of meson theory.

Acknowledgement The authors express their sincere thanks to Professors M. Taketani, S. Nakamura and Y. Nambu for their kind discussion on this work. They are very much indebed to Professor H. Yukawa for informing new results of American authors.

Appendix

Estimation of the Integral I (cf. § 7).

In the integral I, the matrix element

$$w_p(|\boldsymbol{P}_1^0-\boldsymbol{P}_1|)+w_p(|\boldsymbol{P}_1^0-\boldsymbol{P}_2|)$$

of the nuclear potential can be regarded as a slowly varying function of relative momenta $\boldsymbol{P}_1^0-\boldsymbol{P}_1$ and $\boldsymbol{P}_1^0-\boldsymbol{P}_2$, and is approximated to be constant, $2w_p(\sqrt{M\mu}\,c)$, which corresponds to the value for $\boldsymbol{P}_1^0=\boldsymbol{P}_2^0=0$. Then it is sufficient to consider the integral

$$\underset{\substack{|\boldsymbol{P}_1^0|,\,|\boldsymbol{P}_2^0|\leq P_F \\ |\boldsymbol{P}_1|,\,|\boldsymbol{P}_2|\geq P_F}}{\iiiint}\delta(\boldsymbol{P}_1^0+\boldsymbol{P}_2^0-\boldsymbol{P}_1-\boldsymbol{P}_2)\,\delta\left(\frac{\boldsymbol{P}_1^{02}+\boldsymbol{P}_2^{02}-\boldsymbol{P}_1^2-\boldsymbol{P}_2^2}{2M}+\mu c^2\right).$$

$$=\iint dE_1\,dE_2\,J(E_1,\ E_2);$$

$$J(E_1,\ E_2)=\iiiint d\boldsymbol{P}_1^0\,d\boldsymbol{P}_2^0\,d\boldsymbol{P}_1\,d\boldsymbol{P}_2\ \delta\left(\frac{\boldsymbol{P}_1^2}{2M}-E_1\right)\delta\left(\frac{\boldsymbol{P}_2^2}{2M}-E_2\right).$$

$$\cdot\delta(\boldsymbol{P}_1^0+\boldsymbol{P}_2^0-\boldsymbol{P}_1-\boldsymbol{P}_2)\,\delta\left(\frac{\boldsymbol{P}_1^{02}+\boldsymbol{P}_2^{02}-\boldsymbol{P}_1^2-\boldsymbol{P}_2^2}{2M}+\mu c^2\right);$$

$$J(E_1,\ E_2)=J(E_2,\ E_1).$$

$J(E_1,\ E_2)\ dE_1\,dE_2$ represents the relative probability that the two final nucleons have the energies within E_1, E_1+dE_1 and E_2, E_2+dE_2, respectively.

In order to perform the integral in $J(E_1,\ E_2)$, we go on as follows. At first, we perform the integration over the momenta $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$. Next, if we transform the variables $\boldsymbol{P}_1^0$, $\boldsymbol{P}_2^0$ into the new ones

$$\frac{1}{2}(\boldsymbol{P}_1^0+\boldsymbol{P}_2^0),\quad(\boldsymbol{P}_1^0-\boldsymbol{P}_2^0),$$

further integrations can be performed in an elementary way and we find the following result, for $E_1>E_2$

$$J(E_1,\ E_2)=(\text{const.})\times$$

$$\{2\sqrt{E_1E_2}-(\mu c^2-E)\}_+^{3/2}\ \text{ for }\ \frac{1}{2}(\sqrt{\mu c^2}+\sqrt{E})^2>E_1>E_+, \qquad (1)$$

$$[\{2E-(\sqrt{E_F}-\sqrt{E-E_F})^2\}^{3/2}+3(2E_F-E)\}\sqrt{E_F}+\sqrt{E-E_F}-\sqrt{E_1}+\sqrt{E_2}\}],$$
$$\text{for}\quad E_+>E_1>E_-,$$
$$[\{2E-(\sqrt{E_F}+\sqrt{E_F-E})^2\}^{3/2}-\{2E-(\sqrt{E_F}-\sqrt{E_F-E})^2\}^{3/2}$$
$$+\{2\sqrt{E_1E_2}-(\mu c^2-E)\}^{3/2}+6(2E_F-E)\sqrt{E_F}],$$
$$\text{for}\quad E_->E_1>\frac{E+\mu c^2}{2}, \tag{3}$$

where

$$E=E_1+E_2-\mu c^2,$$
$$E_\pm=\frac{1}{2}\left\{E+\mu c^2+\sqrt{(E+\mu c^2)^2-\{\mu c^2\mp 2\sqrt{E_F(E-E_F)}\}^2}\right\}.$$

Finally, the integrations over E_1 and E_2 give the desired result. This result is practically equal to the one obtained for the one when we omit averaging over the Fermi distribution

$$\iint\frac{d\boldsymbol{P}_1^0\,d\boldsymbol{P}_2^0}{\left(\frac{4\pi}{3}\;P_F^3\right)^2}$$

and we put $\boldsymbol{P}_1^0=\boldsymbol{P}_2^0=0$, $P_F=0$.

References

1) H. Fukuda and G. Takeda, Prog. Theor. Phys. **5** (1950), 800; Y. Fujimoto, K. Nishijima, T. Okabayashi, K. Takayanagi and Y. Yamaguchi, Prog. Theor. Phys. **5** (1950), 870.
2) E. Fermi and E. Teller, Phys. Rev. **72**(1947), 399.
3) R. Bjorklund, W. E. Crandall, B. J. Moyer and H. F. York, Phys. Rev. **77** (1950), 213.
4) W. K. H. Panofsky, private communication to Prof. Taketani.
5) F. L. Adelman and S. B. Jones, Phys. Rev. **75** (1949), 1468 (A).
F. L. Adelman, Phys. Rev. **78** (1950), 86 (A).
W. B. Cheston and L. Goldfarb, Bull. Am. Phys. Soc. **25**, No. 1 (1950), 12.
J. Heidemann and L. Leprince-Ringuet, C. R. **226** (1948), 1716.
D. H. Perkins, Phil. Mag. **40** (1949), 601.
6) S. Ogawa and E. Yamada, Prog. Theor. Phys. **5** (1950) 503 (L).
R. E. Marshak and A. S. Wightman, Phys. Rev. **76** (1949), 114.
7) H. Yukawa and S. Sakata, Proc. Phys-Math. Soc. Japan **19** (1937), 712.
B. Bruno, Ark. f. Mat. Astr. e. Fys. **36A**, No. 8(1937), 712; Ark. f. Fys. **1**, No. 2(1949). More complete references will be found in these papers.
8) G. F. Chew and M. L. Goldberger, Phys. Rev. **77** (1950), 470.
9) Y. Fujimoto, S. Hayakawa and Y. Yamaguchi, Prog. Theor. Phys. **4** (1949), 575 (L) and 576 (L).
10) L. L. Foldy and R. E. Marshak, Phys. Rev. **75** (1949), 1493.
11) H. Bradner, *Review of work on artifitially produced mesons*, Oct. 1949.
12) Y. Fujimoto, K. Takayanagi and Y. Yamaguchi, Prog. Theor. Phys. **5** (1950), 498 (L).
13) H. A. Bethe, Phys. Rev. **76** (1949), 191 (A).
14) H. Fukuda and Y. Miyamoto, Prog. Theor. Phys. **4** (1949), 347.
J. Steinberger, Phys. Rev. **76** (1949), 1180.
15) H. Fukuda S. Hayakawa and Y. Miyamoto, Prog. Theor. Phys. **5** (1950), 233, 352.
S. Sasaki, S. Ôneda and S. Ozaki, Prog. Theor. Phys. **5** (1950), 25.

Progress of Theoretical Physics, Vol. 5, No. 6, November~December, 1950

Interpretation of the Second Maximum of Rossi Curve

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(Received May 26, 1950)

§ 1. Introduction

Since long ago, it has been obscure whether the second maximum of Rossi curve really exists or not. If the maxim .m exists, it is very important to explain what kind of showers gives rise to such a maximum. Bothe and others ascertained this phenomenon by their experiment and attempted to explain it as due to hard or knock-on showers.[1] However, Janossy[2] maintained it from his experiment as a spurious effect, while, on the other hand, there exists some experiments[3] which contradict with his result. Although the experiments and the interpretation of the above authors are partly convincing, they seem to be not free from such ambiguity and inconsistency that we are forced to take up this problem on the ground of the later development of cosmic ray physics.

Recent experiment carried out by Kameda and Miura[4] seems to establish the evidence of the existance of the second maximum, and they inferred that this maximum is caused by the nucleonic component on the absorption law of agent rays and of the initial increase of the shower frequencies in lead and paraffin. The similar result was also obtained by Clay[5], but he attributed it to knock-on showers. His interpretation may, however, not be accepted because of the following reasons:

1) Primary rays show the characteristic feature of nucleons as already seen.
2) According to the theoretical and experimental[6] reasons, the saturation of knok-on showers should take place in the much smaller thickness of the absorber of the second maximum.
3) The frequency of the second maximum can not be explained consistently by knock-on hypothesis for both narrow and wide zenith angle of the agent rays, as discussed in later section (§ 4).
4) The absorption coefficient of meson is so small that the knock-on shower can not produce such a sharp maximum.

In this paper, first we describe briefly the experiment of Kameda and Miura (§ 2), and present the further argument for their nucleonic hypothesis (§ 3). Then we show that this assumption gives right order of the frequency of the shower and inquire some conditions to give rise to the appreciable maximum (§ 4). We also discuss some feature of the secondary rays referring to the shape of

the transition curve and study whether or not this feature favours other experimental results (§ 5). But our interpretation contain some ambiguity and may not be conclusive (§ 6).

§ 2. Summary of experimental results

In this section, we give an outline of experimental results obtained by Kameda and Miura. The apparatus is sown in Fig. 1, where 1, 2, 3, 4, 5, 7, 8, are used for coincidences and 6 for anti-coincidence.

The role of the trays 1, 2 and 6 makes essentially their experiment prefer to others. The anti-coincidence (12–6) selects only vertical primary rays and can avoid some ambiguity caused by local and air showers.

We denote the frequency of the various types of showers by the following abbreviation.

NS, (1234—6): Narrow angle showers produced in $\sum_1$ without $\sum_0$.

NPS, (12347—6): NS accompanied by the discharge of at least two counters in tray 7. This was measured for $\sum_2=10$cm Pb.

A, (1234—6): absorption curve of shower producing rays, measured for $\sum_1=10$cm Pb and 17cm Pb. Hereafter, we denote them by A_{10} and A_{17}.

The experimental results of NS and NPS are shown in Fig. 2, where the contribution of knock-on showers is estimated by Kameda and Miura making use of the data of Brown et al[5].

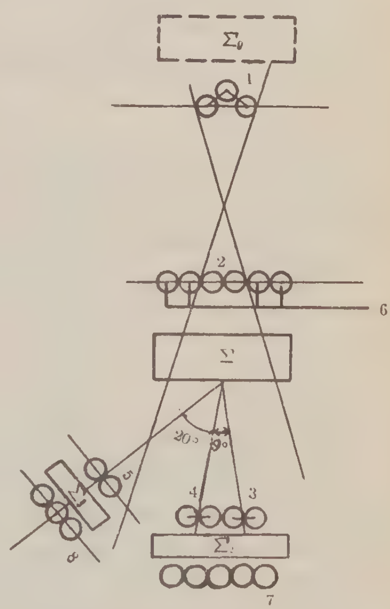

Fig. 1. Counter arrangement

§ 3. Feature of primary rays

Kameda and Miura have already pointed out that the second maximum is caused by nucleonic component on the following two bases.

(1) Subtracting the contribution of knock-on showers, the shape of curve A in Figure 2 can be represented by $e^{-\lambda x}$, where $1/\lambda \sim 15$cm Pb. This figure of collision mean path is equal to that of nucleons obtained by Cocconi et al[7].

Although this interpretation, of course, can not be free from ambiguity caused by the estimation of the knock-on level in Fig. 2, this absorption coefficient will not be far from reality, since both A_{10} and A_{17} are converging to the estimated level.

(2) The initial increase of NPS is far steeper in paraffin than in lead, and

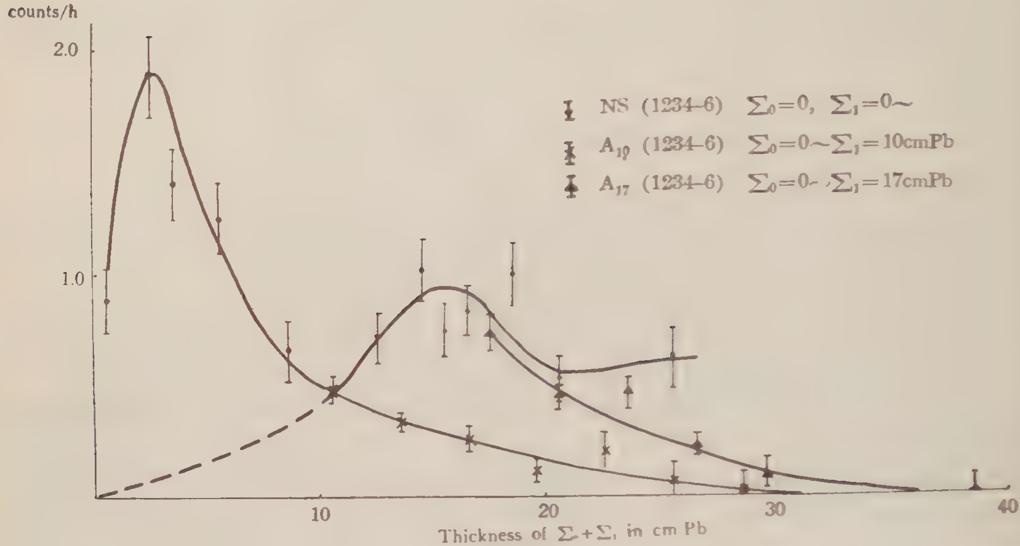

Fig. 2. Transition and absorption curves, contribution of knock-on showers being subtracted.

its material dependence is about $A^{1/3}$. This means that NPS is caused by nucleons.

These evidences seem to support strongly the nucleonic hypothesis, but the relation between the second maximum and NPS is not clear at once. To make this relation clear, we classify NS (only for the case $\sum_1 \geq 10$cm Pb) into two types, as seen in Fig. 3.

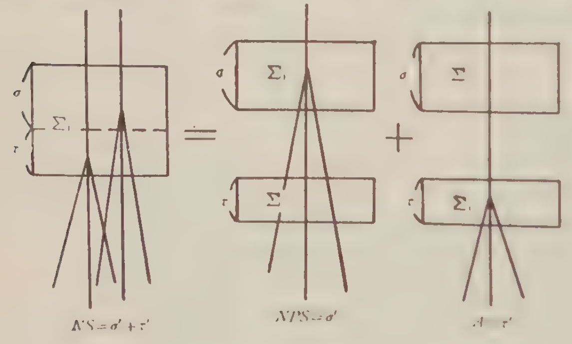

Fig. 3. Decomposition of showers into two types.

We first separate the shower producing material into two parts, i.e. $\tau(=10\text{cm})$ and σ cm, and denote the shower generated in each part by τ' and σ' types respectively, where σ' type, of course, does not involve the shower which can not penetrate through τ.

According to these definitions, NS($\sum_1 \geq 10$cm Pb) consists of $\sigma'(\sigma=\sum_1-\tau)$ and $\tau'(\tau=10\text{cm})$ types, while NPS($\sum_1=\sigma$, $\sum_2=\tau=10$cm Pb) contains only σ' type and $A_{10}(\sum_0=\sigma$, $\sum_2=\tau=10$cm Pb) contains only τ' type.

These relation can be shown by the following schemes:

$$NS=\sigma'+\tau',$$

$$NPS=\sigma',$$

$$A_{10}=\tau',$$

and then

$$\mathrm{NPS}(\Sigma_1=\sigma)=\mathrm{NS}(\Sigma_1-\tau=\sigma)-\mathrm{A}_{20}(\Sigma_0=\sigma).$$

This equality must be held exactly, provided we ignore the detection probability of counter trays. NS−A and NPS can be obtained from the data in Fig. 2 and they are compared in Fig. 4.

The agreement can be said as fairly good, considering the difference of

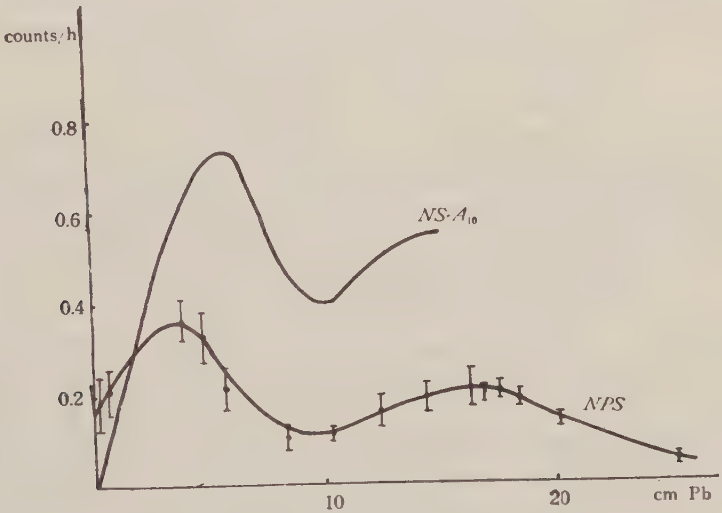

Fig. 4. Comparison of NPS with NS−A_{10}

geometrical conditions. The relation between NPS and NS is now evident, the second maximum of the latter corresponding to the first maximum of the former. Then if we want to know the character of the second maximum, we have only to study the character of the first maximum of NPS.

§ 4. Frequencies of showers

The frequency of the radiation producing the second maximum is about 1 h^{-1} in our case, and this can also be explained consistently by nucleonic primaries. As already pointed out, Clay's interpretation is not appropriate in this case.

Putting the directional intensity of primaries as $j_\perp \cos^n\theta$, the intensity of incoming rays in the circular cone with vertical angle δ is represented by

$$j(\delta)=2\pi\int_0^\delta d\theta\, j_\perp \cos^n\theta \sin\theta=\frac{2\pi j_\perp}{n+1}(1-\cos^{n+1}\delta). \tag{1}$$

As δ is sufficiently small in our case, (1) can be reduced to

$$j(\delta)=\pi j_\perp \delta^2, \tag{2}$$

and it follows

$$j_N(\vartheta)/j_E(\vartheta)=j_{\perp N}/j_{\perp E}, \quad j_N(\vartheta)/j_H(\vartheta)=j_{\perp N}/j_{\perp H}.$$

The values of right hand sides are given by Rossi[8] as

$$j_{\perp N}/j_{\perp E}\sim 1/50, \quad j_{\perp N}/j_{\perp H}\sim 1/200. \tag{4}$$

We put here $j_\perp \cos^n\theta$ for nucleonic, hard, or electronic component as $j_{\perp N}\cos^8\theta$, $j_{\perp H}\cos^2\theta$, or $j_{\perp E}\cos^2\theta$ and, their momentum ranges of these components are taken following Rossi[8]:

nucleonic component : $\gtrsim 10^9$eV/c,

hard ,, : $\gtrsim 3\times 10^8$eV/c,

electronic ,, : $\gtrsim 10^7$eV/c.

From (3), (4) and the total intensity* (coincidence of counters 1, 2) obtained by Kameda and Miura, we can estimate the intensity of each component as follows.

$$j_N \sim 6.10^2 j_{\perp N}/(j_{\perp E}+j_{\perp H})\sim 3h^{-1}, \tag{5}$$

$$j_E \sim 6.10^2 j_{\perp E}/(j_{\perp H}+j_{\perp N})\sim 120h^{-1}, \tag{6}$$

$$j_H \sim 6.10^2 j_{\perp H}/(j_{\perp} +j_{\perp E})\sim 480h^{-1}. \tag{7}$$

From (5), it is plausible to assume that only nucleons with momenta larger than 2BeV contibute to the second maximum, because the actual frequency of the maximum is about $1h^{-1}$. For the first maximum it appears at the thickness of about 2.5cm Pb. The electrons capable to contribute to it, therefore, must have energy larger than $20E_j$, where $E_j=7$MeV is the critical energy for lead.

In order to estimate the intensity of the electrons with energy larger than 140MeV, we tentatively take the integral spectrum of electron $F(E)$ as[5]

$$F(E)\alpha E^{-1} \text{ for } E<50\text{MeV}, \qquad \alpha E^{-2} \text{ for } E>50\text{MeV}, \tag{8}$$

then

$$j \text{ cascade}=F(140\text{MeV})\sim 5h^{-1}. \tag{9}$$

This roughly agrees with the frequency of the first maximum.

Next, we discuss a difficulty of Clay's interpretation. From his experimental results, the intensity of the rays producing the second maximum is about 10^{-4} times of that of the incoming hard component. If the maximum were caused by knock-on showers, this ratio would have to be constant for both narrow and wide angle primaries, but it is about 10^{-2} in our case, so that his presumption can not be accepted.

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* This amounts about $6\cdot 10^2\ h^{-1}$.

Our interpretation, however, is also valid in his experiment, as shown in what follows. As his apparatus selects wide angle primaries, the ratio of the hard to the nucleonic component can be estimated by referring to the relation (1) and (4) as

$$\frac{\text{Hard component}}{\text{Nucleonic component}}=\frac{j_{\perp H}/3}{j_{\perp N}/9}\sim 6\times 10^2, \tag{10}$$

and if we take only nucleons with momenta larger than 2BeV/c, this ratio becomes $2\cdot 10^3$. Considering the contribution of knock-on showers, the value 10^4 can be reduced to the same order as the above estimated value.

Next we consider the reasons why the second maximum distinctly appears in our experiment. In our apparatus, the solid angle of the counter train is very small, which gives much larger values of j_N/j_E than for wide angle primaries. This seems to be the reason why this maximum appears distinctly in our case. If we took wide angle primaries, the tail of the cascade shower and the back ground of knock-on electrons would mask the second maximum.

In order to make sure of such considerations, it is desirable to attempt the following experiments:

(1) The experiment with different solid angles, because j_N/j_E varies with the solid angle.

(2) The experiment at different altitudes, because j_N/j_E varies with altitude.

§ 5. Features of secondary rays

The composition of secondary rays is supposed to largely affect the characteristic shape of the transition curve. As already pointed out, it is sure that the second maximum of NS is shifted to the first maximum of NPS, and then it is plausible to assume that such a maximum appears only at the position of $\sum_1+\sum_2=$constant$\sim$15cm Pb*. Although it is not clear whether or not the second maximum of NPS is the same type as above mentioned, the experimental evidence[9] for the maximum of penetrating showers (this corresponds to NPS for $\sum_2\sim$50cm Pb) at the position $\sum_1\sim$15cm Pb makes us infer these two kinds of the maxima as the same ones, which appear at the thickness of $\sum_1\sim$15cm Pb**.

Then the nature of secondary particles which produce such a maximum can easily be explained as follows. Putting the thickness of $\sum_1$ and $\sum_2$ as x_1 and x_2 (see Fig. 5) respectively, we represent the absorption curves of primary and secondary rays as $e^{-\lambda x}$ and $g(x)E(R-x)$, where $E(x)$ has values 1, 1/2, or 0 for x positive, zero or negative and R means the maximum range of secondaries. The frequency of NPS can be expressed as

* Hereafter, we denote this case as case 1.

**We denote this case as case 2.

$$N(x_1)\sim\int_0^{x_1}dx\, e^{-\lambda x}g(x_1+x_2-x)E(R-x_1-x_2+x). \tag{11}$$

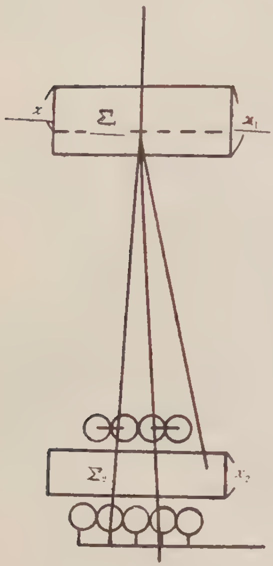

Fig 5. Sketch of penetrating shower.

Transforming the integral variable, (11) is reduced to

$$N(x_1)\sim e^{-\lambda(x_1+x_2)}\{F(x_1+x_2)-F(x_2)\}, \tag{12}$$

where

$$\begin{aligned}F(x)&=\int_0^x e^{\lambda y}g(y)E(R-y)dy\\&=H(x)E(R-x)+H(R)E(x-R),\end{aligned} \tag{13}$$

and

$$H(x)=\int_0^x e^{\lambda y}g(y)dy. \tag{14}$$

We then get the expressions for $N(x_1)$ corresponding to three different ranges of R:

$$N(x_1)\sim 0 \qquad \text{for } R\leq x_2, \tag{15a}$$

$$N(x_1)\sim e^{-\lambda(x_1+x_2)}\{H(R)-H(x_2)\} \qquad \text{for } x_2\leq R\leq x_1+x_2, \tag{15b}$$

$$N(x_1)\sim e^{-\lambda(x_1+x_2)}\{H(x_1+x_2)-H(x_2)\} \qquad \text{for } x_1+x_2\leq R. \tag{15c}$$

For (15b), $N(x_1)$ is a decreasing function, so that the maximum appears at the minimum value of x_1, i.e.

$$x_{1\max}=R-x_2. \tag{16b}$$

For (15c), the maximum position depends on the function H, but in any case

$$x_{1\max}\leq R-x_2. \tag{16c}$$

Referring to these relations, we study the character of function $g(x)\times E(R-x)$ for respective cases.

Case 1. In this case the maximum appears only at the thickness of $x_1+x_2=\text{constant}\sim 15\text{cm}$ Pb, and accounting for (16b) and (16c) we can conclude $x_{1\max}+x_2$ must be equal to R.

This condition is, of course, restricted by the functional form of H.

As a simple example, we put

$$g(x)=e^{-\mu x},$$

then

$$H(x)=\frac{1}{\lambda-\mu}(e^{(\lambda-\mu)x}-1), \tag{14'}$$

and for $x_1+x_2\leq R$

$$N(x_1)\sim\frac{1}{\lambda-\mu}e^{-\mu x_2}\left(e^{-\mu x_1}-e^{-\lambda x_1}\right). \tag{15c'}$$

This gives

$$x_{1\max}=\frac{1}{\lambda-\mu}\ \mathrm{lu}\left(\frac{\lambda}{\mu}\right) \quad \text{provided } \frac{1}{\lambda-\mu}\ \mathrm{lu}\left(\frac{\lambda}{\mu}\right)\leq R-x_2,$$

$$x_{1\max}=R-x_2 \qquad \text{provided } \frac{1}{\lambda-\mu}\ \mathrm{lu}\left(\frac{\lambda}{\mu}\right)\geq R-x_2.$$

Since $x_{1\max}=R-x_2$, $ln(\lambda/\mu)/(\lambda-\mu)$ must be larger than R. This means that the absorption of secondary particle is mainly due to the ionization, and that almost of them have the ranges about 15cm Pb.

Case 2. In this case, the maximum appears only at the position $\sum_1 =\text{constant}\sim 15\text{cm Pb}$. In the limiting case of $x_2=\infty$, we must take $R=\infty$. Hence, $N(x_1)$ is always given by (15c).

Differentiating it by x_1, $x_{1\max}$ is given by the relation

$$\lambda-\lambda\frac{H(x_1+x_2)}{H(x_2)}+\frac{H'(x_1+x_2)}{H(x_2)}=0. \tag{17}$$

As $x_{1\max}$ does not depend on x_2, $H(x)$ must be an exponential function as easily seen from (17). This leads to $g(x)$ of exponential type, which means that the secondary particles responsible to the maximum have high energy.

Summarizing these two results, we may conclude that the second maximum of NS consists of the overlap of two kinds of maxima, mainly due to the particles with definite range. This character of secondaries seems to be favourable to explain some features of the second maximum.

(1) From the general consideration of the shower curve, it can be concluded that the decrease after maximum must be slower than that of the primaries. Only in our case, that the secondaries have a definite range, it is equal to that of the primaries. The experimental decrease is too steep to explain its behaviour by any other absorption law of the secondaries.

(2) The second maximum appears at about the same thickness (in g/cm^2) for various materials[8].

This seems to suggest that the absorption of the secondaries is mainly due to ionization, which results in a definite range.

§ 5. Difficulties of our interpretation

The above interpretation can explain various features of transition curve, but seems to contain some difficulties.

1)* The frequency of showers produced by a single act is generally repre-

* This defect was first pointed out by Dr. Y. Sekido.

stented by $|e^{-\lambda x}-e^{-\mu x}|$, where λ and μ means the absorption coefficients of primaries and secondaries respectively.

This function is convex for the small values of x, whereas the shape of the initial increase of NS is concave, as represented in Fig. 2 by dotted curve. If this experimental result be correct, NS must mainly consist of the showers produced by two or more times of collisions. Here one should note that the statistical error may not be small enough to discuss such a detailed point.

2) The decrease after the maximum of NPS is much steeper than that expected from our presumption.** The experimental data, however, are not so accurate that this defect may also be due to the statistical error of the data.

3) The assumption that the penetrating showers have two kinds of secondaries has already pointed out by Walker[10], but the range of shorter secondaries obtained by him is much shorter than that of ours:

in our case $R\sim 15$cm Pb,

in his case $R\sim 1$cm Pb.

Although the first maximum of NPS can also be explained by considering the detection probability of counters (3 and 4) making use of his result, the relation between this and the second maximum of NS becomes obscure. This also seems to mean that the detection probability does not play an essential role in our case. If we adopt the momenta of primaries as larger than 2BeV/c, which is plausible as the lower limit from the absolute intensity, a shower contains two or more fast particles capable to penetrate through the 10cm lead[11]. Among these particles about a half may be protons which mainly undergo ionization loss. Our interpretation is, therefore, supposed to be not far from reality, and these considerations will be testified by the experiment varying the thickness of $\sum_2$.

The present authors express their hearty gratitude to Messrs. Kameda and Miura who kindly gave us their unpublished data and contributed to our work by valuable advices.

**According to our presumption, it must be slower than that of primaries.

References

1) K. Schmeiser and W. Bothe, Ann. d. Phys. **32** (1938), 161.
W. Bothe, Rev. Mod. Phys. **11** (1939), 282.
A. Ehmert, ZS. f. Phys. **112** (1939), 234.
2) E. P. George, L. Janossy and M. Mclay, Proc. Roy. Soc. A **180** (1942), 219.
3) For example, Z. Ozorai, ZS. f. Phys. **122** (1943), 28.
4) T. Kameda and I. Miura, Prog. Theor. Phys. **5** (1950), 323.
5) J. Clay, Rev. Mod. Phys. **21** (1949), 82.
6) N. W. Brown, A. S. Mckay and E. D. Palmatier, Phys. Rev. **76** (1949), 506.
7) G. Cocconi, Phys. Rev. **75** (1948), 1974; **76** (1948), 984.
8) B. Rossi, Rev. Mod. Phys. **20** (1948), 537.
9) E. P. George and A. C. Jason, Nature **160** (1947), 327
10) W. D. Walker, Phys. Rev. **77** (1950), 686.
11) Y. Fujimoto and S. Hayakawa, Prog. Theor. Phys. **5** (1950), 315.

Progress of Theoretical Physics, Vol. 5, No. 6, November~December, 1950

The Multiple Production of Mesons by High Energy Nucleon-Nucleon Collisions

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(Received June 1, 1950)

§ 1. Introduction

In recent experiments with photographic emulsions, some direct evidences[1] were obtained for the multiple production of mesons by the impacts of very high energy nucleons. It seems to us that these phenomena concerning high energy nucleon-nucleon collisions may offer very interesting informations about the properties of mesons, so, in this paper, we shall treat this problem theoretically. Though there remain some ambiguities on account of the defects in the current field theory and of the strong interaction between nucleons and mesons, we shall treat this problem by perturbation method in the light of Tomonaga-Schwinger relativistic covariant theory.

As the model for mesons, we assume the neutral (or symmetrical) scalar (or pseudoscalar) meson theory, and as the mode of the couplings between nucleons and mesons, we take the g-coupling which contains the derivatives of the meson field. The other type of coupling (f-coupling) has only small contributions to the multiple meson production, as is illustrated in Appendix for the case of neutral vector meson with vector coupling. Further we assume that the momentum transfer between nucleons in collision is performed through the nuclear potential which we take properly and not through the exchange of virtual mesons. Under these assumptions we discuss the multiplicity of mesons, the ratio of the numbers of neutral mesons to the total numbers of emitted mesons and other points. Our attempt is generally the same with the extensive theoretical study by American authors, but is amended in the following two points. That is, here nucleons are treated relativistically throughout and the conservation of charge is taken into account. These two differences will become clear in the course of our calculation.

§ 2. Formula for cross-section

The fundamental wave equation for the system composed of nucleons and mesons is given by

$$\left\{H(x)+V(x)+\frac{1}{i}\frac{\delta}{\delta C}\right\}\Psi[C]=0 \tag{2.1}$$

where $H(x)$ and $V(x)$ are the interaction Hamiltonian density between the both field and the density of the potential between nucleons respectively. Then we make the following canonical transformation for the state vector:

$$\Psi[C]=T[C]\,\Phi[C], \tag{2.2}$$

with

$$\left\{H(x)+\frac{1}{i}\frac{\delta}{\delta C}\right\}T[C]=0 \tag{2.3}$$

This is the relativistic generalization for the Bloch-Nordsieck's canonical transformation,[3] and the wave equation for $\Phi[C]$ becomes

$$\left\{\langle V(x)\rangle+\frac{1}{i}\frac{\delta}{\delta C}\right\}\Phi[C]=0, \tag{2.4}$$

where

$$\langle V(x)\rangle=T^{-1}\cdot V(x)\cdot T. \tag{2.5}$$

$\langle V(x)\rangle$ is the nuclear potential which is modified by the emissions and the absorptions of mesons and gives directly the transition matrices for the multiple meson production. As $V(x)$ is made up by nucleon's wave functions and its adjoint operators, so $\langle V(x)\rangle$ is made up by their transformation. Then, with this $\langle V(x)\rangle$, the total cross section for meson production can be written as follows:

$$\sigma=\frac{(i|\langle V(x)\rangle\int_{-\infty}^{\infty}dx'\langle V(x')\rangle|i)}{(\rho_1/E_1)\cdot(\rho_2/E_2)\cdot B} \tag{2.6}$$

with

$$B=\sqrt{(P_1\cdot P_2)^2-(P_1)^2(P_2)^2}. \tag{2.7}$$

Here (P_1) or (P_2) is the energy-momentum four vector $(P_4=iE)$ of the initial particles 1 or 2, and ρ_1 or ρ_2 is the flux density of 1 or 2. $(i|\langle V(x)\rangle\int dx'\langle V(x')\rangle|i)$ is the diagonal matrix-element of $\langle V(x)\rangle\int dx'\langle V(x')\rangle$ for the initial state.

§ 3. Canonical transformation for the neutral meson field

We shall examine the results of the canonical transformation (2.2) for the neutral meson field.

(A) *Neutral Scalar Meson*

The interaction Hamiltonian density between nucleons and mesons is given by

$$H(x)=\frac{g}{\mu_0}\,\bar{\psi}(x)\gamma_\mu\psi(x)\cdot\frac{\partial\phi(x)}{\partial x_\mu}. \tag{3.1}$$

$\psi(x)$ and $\bar{\psi}(x)$ are the nucleon's wave function and its adjoint function and $\phi(x)$ is that of meson. g is the dimensionless coupling constant, μ_0 is the reciprocal compton wave length of the meson and $\gamma_i=-i\beta\alpha_i$ $(i=1,2,3)$, $\gamma_4=\beta$.

After the canonical transformation $\psi(x)$ and $\bar{\psi}(x)$ are transformed into

$$\langle\psi_\alpha(x)\rangle=\sum_{n=0}^{\infty}\left(\frac{g}{\mu_0}\right)^n\int_{-\infty}^{\infty}dx_1\cdots\int_{-\infty}^{\infty}dx_n\left(\frac{\partial\phi^1}{\partial x_1}\right)_{\mu_1}\left(\frac{\partial\phi^2}{\partial x_2}\right)_{\mu_2}\cdots\left(\frac{\partial\phi^n}{\partial x_n}\right)_{\mu_n}$$

$$\times\{\bar{S}(x-x_1)\gamma_{\mu_1}\bar{S}(x_1-x_2)\gamma_{\mu_2}\cdots\gamma_{\mu_{n-1}}\bar{S}(x_{n-1}-x_n)\gamma_{\mu_n}\psi(x_n)\}_\alpha \tag{3.2}$$

and
$$\langle\bar{\psi}_\alpha(x)\rangle=\sum_{n=0}^{\infty}\left(\frac{g}{\mu_0}\right)^n\int_{-\infty}^{\infty}dx_1\cdots\int_{-\infty}^{\infty}dx_n\left(\frac{\partial\phi^1}{\partial x_1}\right)_{\mu_1}\left(\frac{\partial\phi^2}{\partial x_2}\right)_{\mu_2}\cdots\left(\frac{\partial\phi^n}{\partial x_n}\right)_{\mu_n}$$
$$\times\{\bar{\psi}(x_n)\gamma_{\mu_n}\bar{S}(x_n-x_{n-1})\gamma_{\mu_{n-1}}\cdots\gamma_{\mu_1}\bar{S}(x_1-x)\}_\alpha . \tag{3.3}$$

To obtain (3.2) and (3.3), we used the commutation relations

$$\begin{aligned}\{\psi_\alpha(x),\bar{\psi}_\beta(x')\}&=\frac{1}{i}S_{\alpha\beta}(x-x')\ ,\\ \{\psi_\alpha(x),\psi_\beta(x')\}&=\{\bar{\psi}_\alpha(x),\bar{\psi}_\beta(x')\}=0\end{aligned} \tag{3.4}$$

and symmetrized the integral in past and future by the following equation:

$$\int_{-\infty}^{t_i}dx_j S(x_i-x_j)=-\int_{-\infty}^{\infty}dx_j\,\bar{S}(x_i-x_j). \tag{8.5}$$

Further we disregarded the non-commutability of $\phi(x)$ because of neglecting the reaction of the meson field.

In collision problems we are concerned only with free particles, so it is convenient to proceed with our discussion in momentum representation. $\psi(x)$ and $\bar{\psi}(x)$ are composed of two parts, each of which corresponds to the positive or the negative frequency part, that is, the nucleon or the antinucleon. For the positive frequency part, the Fourier decompositions are

$$\begin{aligned}\psi_\alpha^+(x)&=\frac{1}{\sqrt{2}\,(2\pi)^{3/2}}\sum_{r=1,2}\int\frac{d\boldsymbol{p}}{E_p}\psi_r^+(p)u_{r,\alpha}^+(p)\exp\{ip_\mu\cdot x_\mu\},\\ \bar{\psi}_\alpha^+(x)&=\frac{1}{\sqrt{2}\,(2\pi)^{3/2}}\sum_{r=1,2}\int\frac{d\boldsymbol{p}}{E_p}\bar{\psi}_r^+(p)\bar{u}_{r,\alpha}^+(p)\exp\{-ip_\mu\cdot x_\mu\}\end{aligned} \tag{3.6}$$

where $r(=1,2)$ is the index which indicate the direction of spin. $\psi_r^+(p)$ (or $\bar{\psi}_r^+(p)$) is the annihilation (or creation) operator for the state of positive energy with a momentum $\boldsymbol{p}$ and a direction of spin r.

The commutation relations between them are

$$\{\psi_r^+(p),\bar{\psi}_{r'}^+(p')\}=\delta_{rr'}E_p\delta(\boldsymbol{p}-\boldsymbol{p}'),\ \{\psi_r^+(p),\psi_{r'}^+(p')\}=\{\bar{\psi}_r^+(p),\bar{\psi}_{r'}^+(p')\}=0. \tag{3.7}$$

The spinor wave function $u_{r,\alpha}^+(p)$ and $\bar{u}_{r,\beta}^+(p)$ satisfy the following relation:

$$\sum_{r=1,2}u_{r,\alpha}^+(p)\bar{u}_{r,\beta}^+(p)=(i\gamma_\mu\cdot p_\mu-\varkappa)_{\alpha\beta}\,. \tag{3.8}$$

In the same manner we decompose the meson field operator into Fourier amplitudes. Then we obtain

$$\phi(x)=\frac{1}{\sqrt{2}\,(2\pi)^{3/2}}\int\frac{d\boldsymbol{K}}{\epsilon_k}(\phi^*(K)\exp\{-iK_\mu\cdot x_\mu\}+\phi(K)\exp\{iK_\mu\cdot x_\mu\}). \tag{3.9}$$

$\phi^*(K)$ and $\phi(K)$ are the creation and the annihilation operator of a meson with

momentum $\boldsymbol{K}$ respectively and the commutation relations between them are as follows:

$$[\phi^*(K),\phi(K')]=-\epsilon_k\delta(\boldsymbol{K}-\boldsymbol{K}'),$$
$$[\phi^*(K),\phi^*(K')]=[\phi(K),\phi(K')]=0. \tag{3.10}$$

Now (3.2) can be written as

$$\langle\psi_\alpha^+(x)\rangle=\sum_{n=0}^{\infty}\Big(\frac{1}{\sqrt{2}\,(2\pi)^{3/2}}\Big)^{n+1}\Big(\frac{g}{\mu_0}\Big)^n\sum_{r=1,2}\int\frac{d\boldsymbol{p}}{E_p}\psi_r^+(p)\prod_{i=1}^{n}\int\frac{d\boldsymbol{K}_i}{\epsilon_i}\phi^*(K_i) \tag{3.11}$$
$$_\alpha(p\,;K_n,\cdots,K_1)\exp\{i(p-K_1-K_2\cdots-K_n\cdot x)\}$$

with the use of (3,6), (3.9) and the integral representation $\bar{S}(x)$, that is

$$\bar{S}(x)=\frac{1}{(2\pi)^4}\int d^4l\,\frac{i\gamma\cdot l-\varkappa}{l^2+\varkappa^2}\exp\{i\,l_\mu\cdot x_\mu\}\,. \tag{3.12}$$

In (3.11) U_α is given by

$$U_\alpha(p\,;K_n,\cdots,K_1)=\Big(\frac{i(\gamma\cdot p-K_1-\cdots-K_n)-\varkappa}{(p-K_1-\cdots-K_n)^2+\varkappa^2}\,i\gamma\cdot K_n\,\frac{i(\gamma\cdot p-K_1-\cdots-K_{n-1})-\varkappa}{(p-K_1-\cdots-K_{n-1})^2+\varkappa^2}\cdot$$
$$\times\cdots i\gamma\cdot K_2\frac{i(\gamma\cdot p-K_1)-\varkappa}{(p-K_1)^2+\varkappa^2}\,i\gamma\cdot K_1\cdot u^+(p)\Big)_\alpha. \tag{3.13}$$

We also obtain the similar expression for $\langle\bar{\psi}_\alpha^+(x)\rangle$, in which the conjugate spinor for U_α, that is, $\bar{U}_\alpha$ appears. As is shown in Appendix these quantities satisfy the following relations:

$$\sum_{\text{perm.}}U_\alpha(p\,;K_n,\cdots;K_1)=u_\alpha^+(p), \tag{3.14}$$

and

$$\sum_{\text{perm.}}\bar{U}_\alpha(p\,;K_n,\cdots,K_1)=(-1)^n\bar{u}_\alpha^+(p) \tag{3.15}$$

where $\sum$ means to take the sum of U_α or $\bar{U}_\alpha$ which are obtained by the permutation of $K_1,K_2,\cdots,K_n$. It is to be noted that (3.14) and (3.15) hold without any approximation, and these are related to the divergence theorem for the scalar meson field.

Next we consider the quadratic form in $\psi(x)$ and $\bar{\psi}(x)$ as

$$\bar{\psi}^+(x)\gamma_A\psi^+(x) \tag{3.16}$$

without specializing γ_A. Then after the canonical transformation we get

$$\langle\bar{\psi}^+(x)\gamma_A\psi^+(x)\rangle=\sum_n\Big(\frac{1}{\sqrt{2}\,(2\pi)^{3/2}}\Big)^{n+2}\Big(\frac{g}{\mu_0}\Big)^n\sum_r\sum_{r'_0}\int\frac{d\boldsymbol{p}_0}{E_p}\bar{\psi}_r^+(p)\int\frac{d\boldsymbol{p}_0}{E_{p_0}}\psi_{r_0}^+(p_0)$$
$$\times\Big(\prod_{i=1}^{n}\int\frac{d\boldsymbol{K}_i}{\epsilon_i}\phi^*(K_i)\Big)\,\Gamma_A(p,p_0\,;K_n,\cdots,K_1)\exp\{i(p_0-K_1-\cdots-K_n-p\cdot x)\} \tag{3.17}$$

with

$$\Gamma_A(p, p_0\,; K_n, \cdots, K_1) = \sum_{m=0}^{n} \bar{U}(p\,; K_n, \cdots, K_{m+1})\gamma_A U(p_0\,; K_m, \cdots, K_1). \tag{3.18}$$

By (3.14) and (3.15) we can prove

$$\sum_{\text{perm}} \Gamma_A(p, p_0\,; K_n, \cdots, K_1) = \sum_{m=0}^{n} {}_nC_m(-1)^{n-m}\,\bar{u}^+(p)\gamma_A u^+(p_0) = 0 \tag{3.19}$$

where to take the sum of Γ_A obtained by permuting $K_1, \cdots, K_n$ corresponds to sum up the contributions from the processes which are different with each other in the order of emitting n mesons. Hence the nuclear potential is expressed by the product of two quadratic forms as (3.16), we can conclude from (3.19) that the meson production for neutral scalar mesons does not occur. (This circumstance is due to the divegence theorem).

(B) *Neutral Pseudoscalar Meson*

The interaction Hamiltonian density is

$$H(x) = \frac{-ig}{\mu_0}\,\bar{\psi}(x)\gamma_5\gamma_\mu\psi(x)\frac{\partial\phi(x)}{\partial x_\mu}. \tag{3.20}$$

In this case $\langle\psi^+(x)\rangle$ and $\langle\bar{\psi}^+(x)\gamma_A\psi^+(x)\rangle$ has the same expression as (3.11) and (3.17) respectively, but with different U_α and $\bar{U}_\alpha$. These are

$$U_\alpha(p\,; K_n, \cdots, K_1) = \Big(\frac{i(\gamma\cdot p - K_1 - \cdots - K_n) - \varkappa}{(p - K_1 - \cdots - K_n)^2 + \varkappa^2}\, i\gamma_5\gamma K_n\, \frac{i(\gamma\cdot p - K_1 - \cdots - K_{n-1}) - \varkappa}{(p - K_1 - \cdots - K_{n-1})^2 + \varkappa^2}$$

$$\times \cdots\cdots i\gamma_5\gamma K_2 \frac{i(\gamma\cdot p - K_1) - \varkappa}{(p - K_1)^2 + \varkappa^2}\, i\gamma_5\gamma\cdot K_1\, u^+(p)\Big)_\alpha \tag{3.21}$$

and the corresponding equation for $\bar{U}_\alpha$.

When the incident or the final energy of the nucleon is much larger than both the nucleon mass $\varkappa$ and the energy of mesons emitted by that nucleon, we can obtain approximately the corresponding relations for (3.14) and (3.15) namely

$$\sum_{\text{perm.}} U_\alpha(p\,; K_n, \cdots, K_1) \doteqdot (\gamma_5^{\,n} u^+(p))_\alpha, \tag{3.22}$$

$$\sum_{\text{perm.}} \bar{U}_\alpha(p\,; K_n, \cdots, K_1) \doteqdot (\bar{u}^+(p)\gamma_5^{\,n})_\alpha. \tag{3.23}$$

These derivation will be given in Appendix.

Thus from (3.22) and (3.23), it follows that

$$\sum_{\text{perm.}} \Gamma_A(p, p_0\,; K_n, \cdots, K_1) = \sum_{m=0}^{n} {}_nC_m\,\bar{u}^+(p)\gamma_5^{\,n-m}\gamma_A\gamma_5^{\,m}u^+(p_0)$$

$$= \begin{cases} 2^n\,\bar{u}^+(p)\gamma_5^{\,n}\gamma_A u(p_0) & \text{for } \gamma_A \text{ commuting with } \gamma_5, \\ 0 & \text{for } \gamma_A \text{ anticommuting with } \gamma_5, \end{cases} \tag{3.24}$$

(3.24) shows that only these potentials which have γ_A commuting with γ_5 contribute to the cross section of meson production.*

§ 4. Canonical transformation for the symmetrical meson field

In the symmetrical meson theory positive, negative and neutral mesons may be emitted, but by the charge conservation law the order of emissions of these different kinds of mesons cannot be changed arbitrary. So the results of the canonical transformation (2.2) become complicated. To calculate the transformed quantities we shall make some approximation.

(A) *Symmetrical Scalar Meson Theory*

The interaction Hamiltonian density is

$$H(x)=\frac{g}{\mu_0}\bar{\psi}(x)\tau_i\gamma_\mu\psi(x)\frac{\partial\phi^i}{\partial x_\mu} \tag{4.1}$$

with

$$\tau_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \tau_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \tau_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{4.2}$$

$\phi^1(x)$ ane $\phi^2(x)$ are the wave functions of charged mesons and $\phi^3(x)$ is that of neutral mesons. τ_i** is the isotopic spin operator and the eigenstate $\tau_3=1$ or $\tau_3=-1$ corresponds to the proton or the neutron state. The nucleonic wave function modified by the canonical transformation (2.2) becomes for the positive frequency part

$$\langle\psi_\alpha^+\rangle=\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{2}\,(2\pi)^{3/2}}\right)^{n+1}\left(\frac{g}{\mu_0}\right)^n\sum_{r=1,2}\int\frac{dp}{E_p}\psi_r^+(p)\prod_{i=1}^{n}\int\frac{d\mathbf{K}_i}{\epsilon_i}\phi^{i*}(K_i)$$
$$\times(\tau_i)_n(\tau_i)_{n-1}\cdots(\tau_i)_1\cdot U_\alpha(p\,.\,K_n,\cdots,K_1)\exp i\{p-K_1-\cdots-K_n\cdot x\} \tag{4.3}$$

with the same U_α as (3.13).

Because of the non-commutability of τ_i's between themselves, we cannot obtain such simple relations as (3.15), so we make the following assumptions

$$\begin{aligned}U_\alpha(p\,;K_n,\cdots,K_1)&=\frac{1}{n!}u_\alpha^+(p)\,,\\ \bar{U}_\alpha(p\,;K_n,\cdots,K_1)&=\frac{(-1)^n}{n!}\bar{u}_\alpha^+(p)\,.\end{aligned} \tag{4.4}$$

---

* This result is different from the result obtained by the American authors, because they did not treat the negative virtual states of nucleon correctly.

**The creation (annihilation) operator of a positive (negative) meson K is $\phi^+(K)=(\phi^1(K)-i\phi^2(K))/\sqrt{2}$, $\phi^-(K)=(\phi^1(K)+i\phi^2(K))/\sqrt{2}$, and the charge operator which change a proton (neutron) into a neutron (proton) is

$$\tau_-=\begin{pmatrix}0&1\\0&0\end{pmatrix},\qquad \tau_+=\begin{pmatrix}0&0\\1&0\end{pmatrix}.$$

As is easily understood from (3.14) and (3.15), these relations hold exactly if n mesons are emitted with the same energy-momentum four vectors or will be a good approximation if the energy of emitted meson is not much larger than its rest mass. And in any cases, the permutation of the energy-momentum four vectors between mesons which have the same charges, and the integration with respect to the meson momentum in the last step make the deviation from this approximation smaller.* With these assumptions U_α and $\bar{U}_\alpha$ become independent of the order of $K_i (i=1, 2, \cdots, n)$ and our next task is to calculate the sum of these products $(\tau_i)_n(\tau_i)_{n-1}\cdots(\tau_i)_1$ which are obtained by the permutation of them and are compatible with the charge conservation law.

Consider s mesons of which l are neutral, m positive and n negative and calculate $A_{l,m,n}$. Which is defined by

$$A_{l,m,n.} = (\sqrt{2})^{m+n} \sum_{\text{perm.}} (\tau_3^l \tau_-^m \cdot \tau_+^n) \tag{4.5}$$

$\sum$ means to take the sum of the products of these s isotopic spin operators which are obtained by the permutation of them, and here we do not distinguish same isotopic spin operators among themselves. The values of $A_{l,m,n.}$ are shown in Table I.

Table I. Values of $A_{l,m,n}$

$m=n$	$(\sqrt{2})^{m+n}$	${}_{\left[\frac{s}{2}\right]}C_m\ \tau_3^s$,
$m=n-1$	$(\sqrt{2})^{m+n}$	${}_{\left\{\frac{s-1}{2}\right\}}C_m\ \tau_+$,
$m=n+1$	$(\sqrt{2})^{m+n}$	${}_{\left\{\frac{s-1}{2}\right\}}C_m\ \tau_-$,

where $$\left[\frac{s}{2}\right]=\begin{cases} \dfrac{s}{2} & \text{for even } s, \\ \dfrac{s-1}{2} & \text{for odd } s, \end{cases} \qquad {}_{\left\{\frac{s-1}{2}\right\}}C_m=\begin{cases} 0 & \text{for even } s, \\ {}_{\frac{s-1}{2}}C_m & \text{for odd } s. \end{cases}$$

Next we take up the process in which one of the colliding nucleons emits l neutral, m positive, and n negative mesons.** Some of these $s\,(=l+m+n)$ mesons are emitted before the scattering of the nucleon by the nuclear potential and others are emitted after the scattering. If it produces s_1 (l_1 neutral, m_1 positive and n_1 negative) mesons before the scattering and s_2 (l_2 neutral, m_2 positive and n_2 negative) mesons after the scattering, U_α and $\bar{U}_\alpha$ are proportional to $1/s_1!$ and $(-1)s_1/s_2!$ respectively. Corresponding to this we must obtain the following quantity:

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*The deviation of the matrix element which is evaluated on the assumption (4.4) is very extensive when mesons propagate in the same direction with the nucleon which emits them and $\epsilon/\mu=E/x$. But the contributions from these regions are order of κ^2/E^2 and may be neglected.

**We say one meson is emitted by one of the colliding nucleons if the meson line of that meson is connected with the propagation line of the nucleon in Dyson-Feynman's diagram.

$$B_{l,m,n}(\tau_A)=\sum_{\text{perm.}}\frac{(-1)^{s}}{s_1!\,s_2!}\,(\tau_3^{\,l_2}\tau_-^{\,m_2}\tau_+^{\,n_2}\tau_A\tau_3^{\,l_1}\tau_-^{\,m_1}\tau_+^{\,n_1}). \tag{4.6}$$

Here we assume that the nuclear potential is the product of the quadratic forms in $\bar{\psi}$ and ψ, of which the one concerned with the nucleon now considered has the isotopic spin dependence represented by τ_A. We list the value of $B_{l,m,n}(\tau_A)$ for $\tau_A=1, \tau_3, \tau_-$ or τ_+ in Table II.

Table II. Values of $B_{l,m,n}(\tau_A)$

τ_A	$B_{l,m,n}/(\sqrt{2})^{m+n}$
1	$0,$
τ_3	$\frac{2^s}{s!}\Big[{}_{\{\frac{s}{2}-1\}}C_{m-1}\,\tau_3 A(m,m)-{}_{[\frac{s-1}{2}]}C_m[\tau_- A(m+1,m)+(-1)^s\tau_+ A(m,m+1)]\Big],$
τ_+	$\frac{2^{s-1}}{s!}\Big[(-1)^{s-1}\,{}_{[\frac{s-1}{2}]}C_m\tau_3 A(m+1,m)-{}_{\{\frac{s}{2}-1\}}C_{m-1}\,\tau_- A(m+1,m-1)$ $+(-1)^s\Big\{{}_{[\frac{s}{2}]}C_m+{}_{[\frac{s-1}{2}]}C_m\Big\}\tau_+ A(m,m)\Big],$
τ_-	$\frac{2^{s-1}}{s!}\Big[-{}_{[\frac{s-1}{2}]}C_m\tau_3 A(m,m+1)-{}_{\{\frac{s}{2}-1\}}C_{m-1}\,\tau_+ A(m-1,m+1)$ $+\Big\{{}_{[\frac{s}{2}]}C_m+{}_{[\frac{s-1}{2}]}C_m\Big\}\tau_- A(m,m)\Big],$

(B) *Symmetrical Pseudoscalar Meson*

For the symmetrical pseudoscalar meson theory the interaction Hamiltonian density is

$$H(x)=\frac{-ig}{\mu_0}\bar{\psi}(x)\tau_i\gamma_5\gamma_\mu\psi(x)\frac{\partial\phi^i(x)}{\partial x_\mu} \tag{4.7}$$

and, as in the case (A), we assume

$$U_\alpha(p\,;K_n,\cdots,K_1)=\frac{1}{n!}(\gamma_5^{\,n}u^+(p))_\alpha\,,$$

$$U_\alpha(p\,;K_n,\cdots,K_1)=\frac{1}{n!}(\bar{u}^+(p)\gamma_5^{\,n})_\alpha\,. \tag{4.8}$$

If γ_A represents the spin dependence of the nuclear potential anticommuting with γ_5, we can use Table II also. But if γ_A commutes with γ_5, it becomes necessary to calculate the following expression instead of (4.6) :

$$B'_{l,m,n}(\tau_A)=\sum_{\text{pemr.}}\frac{1}{s_1!\,s_2!}\,(\tau_3^{\,l_1}\tau_-^{\,m_2}\tau_+^{\,n_2}\tau_A\tau_3^{\,l_2}\tau_-^{\,m_1}\tau_+^{\,n_1})\,. \tag{4.9}$$

The results are given in Table III.

Table III. Values of $B'_{l,m,n}(\tau_A)$

	$B'_{l,m,n}(\tau_A)/(\sqrt{2})^{m+n}$
1	$\frac{2^s}{s!}\left[{}_{\left[\frac{s}{2}\right]}C_m\,\tau_3{}^s\,A(m,m)+{}_{\left\{\frac{s-1}{2}\right\}}C_m\{\tau_-A(m+1,m)+\tau_+A(m,m+1)\}\right]$,
τ_3	$\frac{2^s}{s!}\left[{}_{\left[\frac{s-1}{2}\right]}C_m\tau_3{}^{s+1}A(m,m)+{}_{\left\{\frac{s}{2}-1\right\}}C_m\{\tau_-A(m+1,m)+\tau_+A(m,m+1)\}\right]$,
τ_+	$\frac{2^{s-1}}{s!}\left[{}_{\left[\frac{s-1}{2}\right]}C_m\,\tau_3{}^{s+1}\,A(m+1,m)+{}_{\left\{\frac{s}{2}-1\right\}}C_{m-1}\{\tau_-A(m+1,m-1)+\tau_+A(m,m)\}\right]$,
τ_-	$\frac{2^{s-1}}{s!}\left[{}_{\left[\frac{s-1}{2}\right]}C_m\,\tau_3{}^{s+1}\,A(m,m+1)+{}_{\left\{\frac{s}{2}-1\right\}}C_{m-1}\{\tau_-A(m,m)+\tau_+A(m-1,m+1)\}\right]$,

$A(m,n)$ means that the term is connected to the emission of m positive mesons and n negative mesons.

§ 5. Transformed nuclear potential

Using the results of § 3 and § 4 we discuss that part of the transformed nuclear potential which is responsible for the transition of two nucleons from p_0, q_0 to p, q with the emission of L neutral, M positive and N negative mesons. As these mesons are to be emitted by either the one or the other of the two colliding nucleons, we must take account all possibilities dividing them into two groups according to their belonging nucleons.* For this purpose we compute the following summations:

$$(p,q|K^{(s)}_{L,M,N}(\tau_A;\gamma_A)|p_0,q_0)=\sum_{l=0}^{L}\sum_{m=0}^{M}\sum_{n=0}^{N}(p|\gamma_AB_{l,m,n}(\tau_A)|p_0)(q|\gamma_AB_{l',m',n'}(\tau_{A'})|q_0) \tag{5.1}$$

for the scalar theory,

$$(p,q|K'^{(ps)}_{L,M,N}(\tau_A;\gamma_A)|p_0,q_0)=\sum_l\sum_m\sum_n(p|\gamma_5{}^s\gamma_AB'_{l,m,n}(\tau_A)|p_0)(q|\gamma_5{}^{s'}\gamma_AB'_{l',m',n'}(\tau_{A'})|q_0) \tag{5.2}$$

for the symmetrical pseudoscalar theory and the nuclear potential with γ_A commuting with γ_5, and

$$(p,q|K^{(ps)}_{L,M,N}(\tau_A;\gamma_A)|p_0,q_0)=\sum_l\sum_m\sum_n(p|\gamma_5{}^s\gamma_AB_{l,m,n}(\tau_A)|p_0)(q|\gamma_5{}^{s'}\gamma_AB_{l',m',n'}(\tau_{A'})|q_0) \tag{5.3}$$

for the symmetrical pseudoscalar theory and the nuclear potential with γ_A anticommuting with γ_5. Here $l+l'=L$, $m+m'=M$ and $n+n'=N$, and $(p/\gamma_AB_{l,m,n}(\tau_A)/p_0)=(\bar{u}^+(p)\gamma_AB_{l,\cdot\cdot,n}(\tau_A)u^+(p_0))$ etc.

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* Different divisions results in the different momentum transfers between nucleons, but this will be neglected when the energy loss of nucleons in the collision is smaller than their incident energies in the center of the gravity system.

As for the neutral theories, we want confine ourselves to say that only the neutral pseudoscalar theory and the nuclear potential with γ_A commuting with γ_5 can give large contributions to the multiple meson production and in this case the multiplicity, the average angular spread and other properties of emitted meson are almost the same as those in the previous authors.

Hereafter we fix the form of the nuclear potential as follows:

$$V(x)=F^2\left\{\sum_{i=1}^{3}\bar{\psi}(x)\tau_i\gamma_A\psi(x)\int_{-\infty}^{\infty}dx'\bar{\Delta}_{\mu_0}(x-x')\bar{\psi}(x')\tau_i\gamma_A\psi(x')\right.$$

$$\left.+\eta\cdot\bar{\psi}(x)\gamma_B\psi(x)\int_{-\infty}^{\infty}dx'\bar{\Delta}_{\mu_0}(x-x')\bar{\psi}(x')\tau_i\gamma_B\psi(x')\right\}, \tag{5.4}$$

where γ_A and γ_B are left unfixed. This potential is the mixture of the symmetrical and the neutral one with the mixing ratio η, and $\bar{\Delta}_{\mu_0}$ with mass parameter μ_0 represents the $1/r$ dependence of the potential, but this choice of r-dendence will be no limitation for the following discussions.

Then we must calculate for this potential the similar quantities as (5.1), (5.2) and (5.3) employing Table II and III, and the results become as follows: For the scalar meson theory

$$(p,q|\sum_{i=1}^{3}K_{L,M,N}^{(s)}(\tau_i;\gamma_A)|p_0,q_0)\equiv 0$$

and

$$(p,q|K_{L,M,N}^{s}(1;\gamma_A)|p_0,q_0)\equiv 0. \tag{5.5}$$

For the pseudoscalar case the results are shown in Table IV and V corresponding to (5.2) and (5.3) respectively.

As is seen from (5.5), Table IV and V, only the case of symmetrical pseudoscalar meson theory is interesting for us and will be treated in detail in the following sections.

Table IV. Values of $(p,q|\sum_{i=1}^{3}K'^{(ps)}_{L,M,N}(\tau_i;\gamma_A)+\eta K'^{(ps)}_{L,M,N}(1;\gamma_B)|p_0,q_0)/\sqrt{2}^{(M+N)}$

$M=N$

$$\frac{2^{2s-1}}{s!}\left\{\left[{}_{\left[\frac{s-1}{2}\right]}C_M(p|\gamma_A\tau_3|p_0)(q|\gamma_A\gamma_5{}^s\tau_3{}^{s-1}|q_0)+{}_{\left[\frac{s}{2}\right]}C_M(p|\gamma_A\gamma_5|p_0)(q|\gamma_A\gamma_5{}^{s-1}\tau_3{}^s|q_0)\right.\right.$$

$$\left.+2\cdot{}_{\left\{\frac{s}{2}-1\right\}}C_M(p|\gamma_A\tau_+|p_0)(q|\gamma_A\tau_-|q_0)\right]$$

$$+\eta\left[{}_{\left[\frac{s}{2}\right]}C_M(p|\gamma_B|p_0)(q|\gamma_B\gamma_5{}^s\tau_3{}^s|q_0)+{}_{\left[\frac{s-1}{2}\right]}C_M(p|\gamma_B\gamma_5{}^s\tau_3|p_0)(q|\gamma_B\gamma_5{}^{s-1}\tau_3{}^{s-1}|q_0)\right.$$

$$\left.\left.-2\cdot{}_{\left\{\frac{s}{2}-1\right\}}C_M(p|\gamma_B\gamma_5\tau_+|p_0)(q|\gamma_B\gamma_5\tau_-|q_0)\right]+\langle p,q;p_0,q_0\rangle\right\},$$

$M=N\pm 1$, Min.$(M, N)=M'$.

$$\frac{2^{2s}}{s!}\Bigg[{}_{\left[\frac{s-1}{2}\right]}C_{M'}\Big\{(p\,|\,\gamma_A\gamma_5{}^s\tau_3{}^{s+1}\,|\,p_0)(q\,|\,\gamma_A\tau_\pm\,|\,q_0)+\eta(p\,|\,\gamma_B\gamma_5{}^{s-1}\tau_3{}^s\,|\,p_0)(q\,|\,\gamma_B\gamma_5\tau_\mp\,|\,q_0)\Big\}$$
$$+\langle p,q\,;p_0,q_0\rangle\Bigg],$$

$M=N\pm 2$, Min.$(M, N)=M'$

$$\frac{2^{2s-1}}{s!}\Bigg[{}_{\left[\frac{s}{2}-1\right]}C_{M'}\Big\{(p\,|\,\gamma_A\tau_\mp\,|\,p_0)(q\,|\,\gamma_A\tau_\mp\,|\,q_0)+\eta(p\,|\,\gamma_B\gamma_5\tau_\mp\,|\,p_0)(q\,|\,\gamma_B\gamma_5\tau_\mp\,|\,p_0)\Big\}$$
$$+\langle p,q\,;p_0,q_0\rangle\Bigg],$$

$\langle p,q\,;p_0,q_0\rangle$ is the expression which is obtained from the previous one by interchanging p with q and p_0 with q_0 simultaneously.

Table V. Values of $(p,q\,|\sum_{i=1}^{3}K^{(ps)}_{L,M,N}(\tau_i\,;\gamma_A)\,|\,p_0q_0)/\sqrt{2}^{\,(M+N)}$

$M=N$

$$\frac{2^{2s-1}}{s!}\Bigg[{}_{\left\{\frac{s}{2}-1\right\}}C_{M-1}\Big\{(p\,|\,\gamma_A\tau_3\,|\,p_0)(q\,|\,\gamma_A\tau_3\,|\,q_0)-(p\,|\,\gamma_A\gamma_5\tau_3\,|\,p_0)(q\,|\,\gamma_A\gamma_5\tau_3\,|\,q_0)\Big\}$$
$$+2\Big\{{}_{\left[\frac{s}{2}\right]}C_M+{}_{\left[\frac{s-1}{2}\right]}C_M\Big\}\Big\{(p\,|\,\gamma_A\gamma_5{}^s\tau_-\,|\,p_0)(q\,|\,\gamma_A\tau_+\,|\,q_0)-(p\,|\,\gamma_A\gamma_5{}^{s-1}\tau_-\,|\,p_0)\times$$
$$\times(q\,|\,\gamma_A\gamma_5\tau_+\,|\,q_0)\Big\}+\langle p,q\,;p_0,q_0\rangle\Bigg],$$

$M=N\pm 1$, Min.$(M, N)=M'$

$$\frac{2^{2s}}{s!}\Bigg[{}_{\left[\frac{s-1}{2}\right]}C_{M'}\Big\{(p\,|\,\gamma_A\gamma_5{}^{s-1}\tau_3\,|\,p_0)(q\,|\,\gamma_A\gamma_5\tau_\pm\,|\,q_0)-(p\,|\,\gamma_A\gamma_5{}^s\tau_3\,|\,p_0)\times$$
$$\times(q\,|\,\gamma_A\tau_\pm\,|\,q_0)\Big\}+\langle p,q\,;p_0,q_0\rangle\Bigg],$$

$M=N\pm 2$, Min.$(M, N)=M'$

$$\frac{2^{s-1}}{s!}\Bigg[{}_{\left\{\frac{s}{2}-1\right\}}C_{M'}\Big\{(p\,|\,\gamma_A\gamma_5\tau_\pm\,|\,p_0)(q\,|\,\gamma_A\gamma_5\tau_\pm\,|\,q_0)-(p\,|\,\gamma_A\tau_\pm\,|\,p_0)(q\,|\,\gamma_A\tau_\pm\,|\,q_0)\Big\}$$
$$+\langle p,q\,;p_0,q_0\rangle\Bigg].$$

For the neutral nuclear potential

$$(p,q\,|\,K^{(ps)}_{L,M,N}(1\,;\gamma_B)\,|\,p_0,q_0)\equiv 0.$$

§6. The cross-section for the multiple meson production and the ratio of the numbers of neutral mesons to the total numbers

Now substituting $\langle V(x)\rangle$ obtained in §5 into the formula (2.6), the total cross section for the process, in which L neutral, M positive and N negative

mesons are produced and thereby the colliding two nucleons are scattered from p_0, q_0 to p, q becomes as follows:

$$\sigma_{L,M,N.}=\frac{(2\pi)^4}{4B}\left(\frac{1}{2(2\pi)^3}\right)^{s+2}\left(\frac{g}{\mu_0}\right)^{2S}\int\frac{dp}{E_p}\int\frac{dq}{E_q}\prod_{i=1}^{S}\int\frac{d\boldsymbol{K}_i}{\epsilon_i}\int\delta(\sum K_i+p+q-p_0-q_0)$$

$$\times F^4|(p,q|K_{L,M,N}|p_0,q_0)\frac{1}{\mu_0^2+(p-p_0)^2}+\langle p,q;p_0,q_0\rangle-\langle p,q\rangle-\langle p_0,q_0\rangle|^2 \qquad (6.1)$$

where $\langle p,q\rangle$ or $\langle p_0,q_0\rangle$ is the expressions which is obtained from the previous one by interchanging p with q or p_0 with q_0 respectively, and $(p,q|K_{L.M.N}|p_0,q_0)$ is the abbreviation for (5.2) or (5.3). Hereafter we proceed with our discussion in the center of gravity system.* When the incident energy of the nucleon which is the same for the two nucleons in this system is much larger than the energy loss by the emission of mesons, the energy momentum conservation is approximately satisfied only by the two nucleons. For this reason the total cross section (6.1) can be interpreted as the product of two factors, of which the one comes from the scattering of two free nucleons without producing mesons and the other is due to the emission of mesons on account of the difference of the nucleon's eigen fields before and after the scattering. So, at first, we perform the integration in the meson's momentum space, taking into account only the limitation that the total energy of mesons should be equal or smaller than the some fixed value ϵ. About the value we shall make some discussion later. Then

$$\prod_{\substack{i=1\\(\epsilon>\Sigma\epsilon_i)}}^{S}\int\frac{d\boldsymbol{K}_i}{\epsilon_i}=(4\pi)^s\int_{\mu}^{\epsilon}k_s d\epsilon_s\int_{\mu}^{\epsilon-\epsilon_s}k_{s-1}d\epsilon_{s-1}\cdots\int_{\mu}^{\epsilon-\sum_{i=2}^{s}\epsilon_i}k_1 d\epsilon_1$$

$$\sim(4\pi)^s\int_0^{\epsilon}\epsilon_s d\epsilon_s\int_0^{\epsilon-\epsilon_s}\epsilon_{s-1}d\epsilon_{s-1}\cdots\int_0^{\epsilon-\sum_{i=2}^{s}\epsilon_i}\epsilon_1 d\epsilon_1=\frac{(4\pi\epsilon^2)^s}{(2s)!}. \qquad (6.2)^{**}$$

Next we calculate the cross section of the nuclear scattering given by

$$\sigma_0=\frac{(2\pi)^4}{4B}\left(\frac{1}{2(2\pi)^3}\right)^2\int\frac{dp}{E_p}\int\frac{dq}{E_q}\delta(p+q-p_0-q_0)|(p,q|V|p_0,q_0)|^2 \qquad (6.3)$$

where

---

*Only in this system our approximations for U_a and $\overline{U}_a$ can be justified.

**This simple calculation gives the same result as that of the more detailed calculation of Lewis, Oppenheimer and Wouthuysen. We can also prove the following inequality:

$$\frac{(4\pi\epsilon^2)^s}{(2s)!}>\prod_{\substack{i=1\\(\epsilon\geq\Sigma\epsilon_i)}}^{s}\int\frac{d\boldsymbol{K}_i}{\epsilon_i}>\frac{(4\pi(\epsilon-s\mu)^2)^s}{(2s)!}$$

and the exact value is nearer to the upper limit in case of $\epsilon\gtrsim 3s\mu_0$.

$$(p, q|V|p_0, q_0)=\Big[(p|\tau_\alpha\gamma_\alpha|p_0)(q|\tau_\beta\gamma_B|q_0)\frac{1}{\mu^2+(p-p_0)^2}$$
$$+\langle p, q\,;p_0\,q_0\rangle-\langle p, q\rangle-\langle p_0, q_0\rangle\Big] \qquad (6.4)$$

for the transformed potentials which appear in Table IV ard V. Finally we must take the sum of (6.1) for different divisions of s mesons into L, M, N mesons in order to obtain the total cross section for creating s mesons without regard to their charged. The results are given in Table VI, VII with the ratio of neutral mesons.

Table VI. Values of σ_s and R for the potential $\gamma_A=\gamma_5$ (or 1)

Case I. s=odd

initial nucleons	final nucleons	σ_s	R
$p,p(n,n)$	$p,p(n,n)$	$\xi_s\frac{s+2}{3}(1+\eta)^2(8+4\Delta)$	$\frac{3s+2}{5s}$
p,n	p,n	$\xi_s\frac{s+2}{3}(1-\eta)^2 2\Delta$	$\frac{3s+2}{5s}$
$p,p(p,n)$ $n,n(p,n)$	$p,n(p,p)$ $p,n(n,n)$	$\xi_s\frac{s+2}{3}(6+4\eta+6\eta^2+4(1+\eta^2)\Delta)$	$\frac{s-1}{5s}$
$p,p(n,n)$	$n,n(p,p)$	0	—

$$\xi_s=\left(\frac{F^2}{4\pi}\right)^2\frac{\pi}{4E^2}\left(\frac{2f\epsilon}{\pi\mu_0}\right)^{2s}\frac{1}{s!(2s)!},\qquad \Delta=\frac{x^2}{p^2}\left(\log\left(\frac{\mu_0{}^2+4p^2}{\mu_0{}^2}\right)-1\right).$$

Case II. s=even

initial nucleons	final nucleons	σ_s	R
$p,p(n,n)$	$p,p(n,n)$	$\xi_s(s+1)\Big[\left(\frac{3s+4}{15s}+\frac{2}{3}\eta+\eta^2\right)\cdot A+\left(1+\frac{2}{3}\eta+\frac{3s+4}{15s}\eta^2\right)3$ $+\left\{\frac{1}{3}(1+\eta^2)+\eta\left(\frac{18s+4}{15s}\right)\right\}2\cdot B\Big]$	
$p,p(p,n)$ $n,n(p,n)$	$p,n(p,p)$ $p,n(n,n)$	$\xi_s\frac{2(s+1)(s+3)}{15s}[A+2\eta B+3\eta^2]$	$\frac{3s+1}{7s}$
$p,p(n,n)$	$n,n(p,p)$	$\xi_s\frac{8(s+1)(s+3)}{15s}[A+2\eta B+3\eta^2]$	$\frac{s-2}{7s}$
p,n	p,n	$\xi_s(s+1)\Big[\left(\frac{11s+8}{15s}-\frac{2}{3}\eta+\eta^2\right)C+\eta^2\frac{8s+4}{15s}$ $+\left(\frac{2s-4}{15s}-\frac{2}{3}\eta\right)(D-\eta B)+\left(\frac{2}{3}-\frac{2s-4}{15s}\eta\right)(B-\eta)\Big]$	

$$A=3+\frac{8x^4}{\mu_0^2p^2}+\frac{2x^2(p^2-x^2)}{p^4}\log\left(\frac{4p^2}{\mu_0{}^2}\right),\qquad B=1+\frac{x^2}{p^2}\log\left(\frac{4p^2}{\mu_0{}^2}\right),$$
$$C=1+\frac{2x^2}{p^2}\log\left(\frac{4p^2}{\mu_0{}^2}\right)+\frac{4x^4}{\mu^2p^2},\qquad D=2\frac{x^2(x^2+p^2)}{p^4}\log\left(\frac{4p^2}{\mu_0{}^2}\right)-1.$$

Here E and p is the energy and the magnitude of momentum for the nucleon in the center of gravity system, and we neglect the terms of order μ_0^2/κ^2 and μ_0^2/p^2. For the processes $pp \rightarrow pp$ and $p, n \rightarrow p, n$ in case II, R's are very much complicated, and the approximate values for $E \geq 10\kappa$ are given in table VIII. In case II, (F) and $(F\eta)$ are the coupling constants of symmetrical (or neutral) and neutral (or symmetrical) potential with $\gamma_A = \gamma_B = \gamma_5$ (or 1) respectively.

Table VII. Values of σ_s and R for the potential $\gamma_A = \gamma_B$ (or $\gamma_5\gamma_B$)

initial nucleons	final nucleons		σ_s	R
$p, p(n, n)$	$p, p(n, n)$	odd	0	—
		even	$16\xi_s \dfrac{2(s+1)(2s+1)}{15s} X$	$\dfrac{(2s-1)(s-2)}{7s(2s+1)}$
$p, p(n, n)$	$n, n(p, p)$	odd	0	—
		even	$16\xi_s \dfrac{4(s+1)(s+3)}{15s} X$	$\dfrac{s-2}{7s}$
$p, p(p, n)$ $n, n(p, n)$	$p, n(p, p)$ $p, n(n, n)$	odd	$16\xi_s \dfrac{(s+2)}{3} Y$	$\dfrac{s-1}{5s}$
		even	$16\xi_s \dfrac{(s+1)(s+3)}{15s} X$	$\dfrac{3s+1}{7s}$
p, n	p, n	odd	$16\xi_s \dfrac{(s+2)}{5} Y$	$\dfrac{3s+2}{5s}$
		even	$16\xi_s \dfrac{(s+1)(9s+2)}{15s} Z$	$\sim \dfrac{15s+18s+4}{7s(9s+2)}$

$$X = \left[\frac{1}{\mu_0^2}\left(2E^2 + \frac{\kappa^4}{p^2}\right) - \frac{(\kappa^2+2p^2)(p^2-\kappa^2)}{2p^4} \log\left(\frac{4p^2}{\mu_0^2}\right) + \frac{1}{2}\right],$$

$$Y = \left[2\frac{E^2}{\mu_0^2} - \frac{\kappa^2+2p^2}{2p^2} \log\left(\frac{4p^2}{\mu_0^2}\right) + \frac{1}{2}\right], \qquad Z = Y + \frac{\kappa^4}{\mu_0^2 p^2} \qquad \text{and} \quad X \fallingdotseq Y \fallingdotseq Z.$$

§ 7. Discussions

Here we compare the total cross sections for the four different types of mesons which we adopted.

(A) *Neutral scalar meson theory*

The cross section vanishes identically.

(B) *Neutral pseudoscalar meson theory*

The results are the same with those by the previous authors, except that only the nuclear potentials which have γ_A commuting with γ_5 contribute to the meson production.

(C) *Symmetrical scalar meson theory*

As in the case (A) the cross section vanishes exactly if our assumption

(4.4) holds, but the deviation from it gives a small cross section.

(D) *Symmetrical pseudoscalar meson theory*

Because this case has the much larger cross section for multiple meson production compared with other cases, we discuss further in detail the multiplicity of mesons, the neutral meson ratio and some other points.

(1) The multiplicity of mesons

The relation between the total cross section for the multiple meson production and the total number of mesons emitted is given by

$$\sigma_s \propto \xi_s \cdot s. \tag{7.1}$$

From (8.1), the average number of mesons $\bar{s}$ and the half breadth δ of the $\sigma_s - s$ distribution curve when the maximum energy loss ϵ is fixed become as follows:

$$\bar{s} \sim \left(\frac{f\epsilon}{\pi\mu_0}\right)^{2/3} \tag{7.2}$$

and

$$\delta \sim \sqrt{\bar{s}/3}. \tag{7.3}$$

The $\sigma_s - s$ distribution curve for different values ϵ are shown in Fig. 1. ϵ, the maximum energy loss, is to be determined by considering the reaction of the meson field and the applicability of our assumption that the energy loss should be smaller than the incident energy. Without detailed consideration, we assume that ϵ is proportional to the initial energy E, that is,

$$\epsilon = 2 \cdot \gamma \cdot E \tag{7.3}$$

and γ, the proportional constant, is approximately 1/3. Then to obtain $\bar{S}=4$, the incident energy E of the each nucleon in the center of gravity system becomes

$$E \sim 6 \cdot \text{Bev.} \qquad \text{for} \quad f^2 = 1,$$

or

$$E \sim 3 \cdot \text{Bev.} \qquad \text{for} \quad f^2 = 4.$$

(2) The ratio of neutral mesons

The ratio of neutral mesons is shown in Table VI, and VII, but is very complicated for the case of $s=$ even and $\gamma_A = \gamma_5$ or 1. So here we give the approximate value of them in Table VIII for $E \geq 10\varkappa$. Thus R changes according to the charge states of the two nucleons before and after the scattering, the total number of mesons (odd or even) and also the nuclear potential which is assumed. So, for example, to obtain average R which is expected when a high energy proton collides with a nucleus having equal numbers of protons and neutrons, we must average R for various charged states of the nucleons before and after the collision. Then by use of Table VI and VII, it follows approximately $R \sim 1/3$.

(3) Comparison of the total cross section for various processes

In case of γ_A(or 1), the cross section in which odd numbers of mesons are emitted is smaller than that for even number of mesons in low energy collisions ($E < 5\varkappa$). But in high energy collisions ($E > 10\varkappa$), some odd meson processes are as probable as that of even and this is shown in graph 2. In other cases, that is $\gamma_A = \gamma_\mu$ (or $\gamma_5\gamma_\mu$), the cross section for odd number of mesons is the same for even numbers.

The energy dependence of cross section is not reliable on account of our neglection of damping effects. But it is to be noted that in the former case the part of the cross section corresponding to the nuclear scattering without meson production part has the energy dependence $1/E^2$, whereas in the latter case it becomes independent of the incident energy which is sufficiently large.

(4) The angular distribution of mesons and nucleons

The angular distribution of mesons are almost spherically symmetric in the center of gravity system. But the distribution of final nucleons depends on the potential type, i.e., for $\gamma_A = \gamma_5$ (or 1) it is almost spherically symmetric and for $\gamma_A = \gamma_\mu$ (or $\gamma_5\gamma_\mu$) it has sharp maximum for forward and backward directions. (The average angle of deflection $\theta \sim \mu_0/p$.)

Concluding remarks

Our results have some ambiguities not only from the poor knowledge about meson type, nuclear potential responsible for high energy collisions, but also from neglecting the damping and reactive corrections. The following points, however, may be considered as proper conclusions of our calculations.

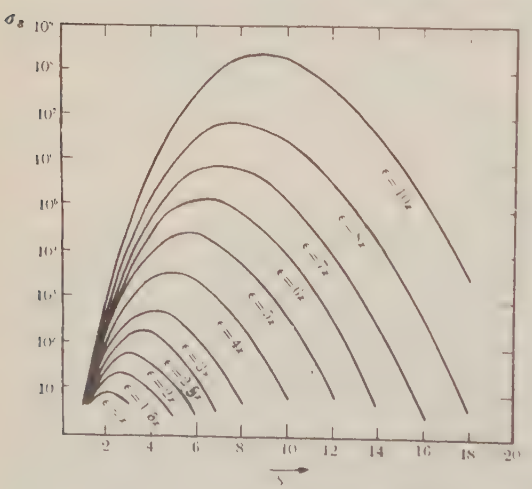
1) Scalar meson may not be the mesons which are emitted multiply in high energy collisions of nucleons.

2) It seems that pseudoscalar mesons with pseudovector couplings are responsible for the high multiplicity of meson production.

3) The experimental indication for forwards angular distribution of final nucleons[4] make probable the existence of vector type nuclear force.

We should express our gratitude to Mr. S. Hayakawa and Mr. Y. Fujimoto for their helpful discussions.

Fig. 1. The $\sigma_s - s$ distribution curve for different values of ϵ ($f^2 \sim 2$)



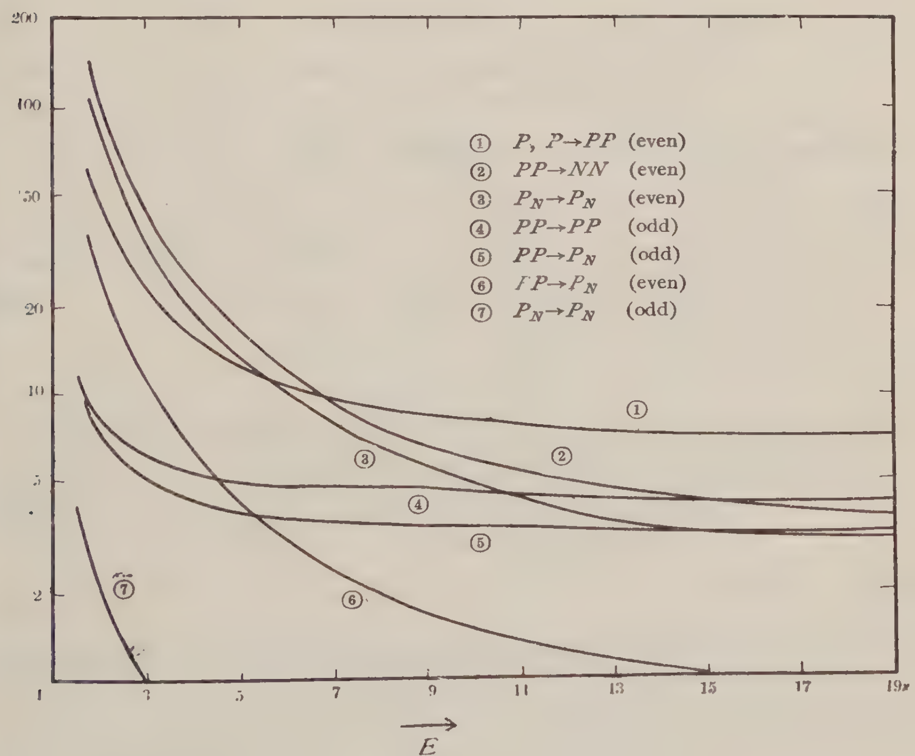
Absolute values have no meaning, considering the damping effects.

Table VIII. Approximate values of the neutral meson ratio R

$(E \geq 10\varkappa (s \gg 1))$

γ_A (or γ_B)	s	V	$p,p-p,p$;	$p,n-p,n$;	$p,p-p,n$;	$p,p-n,n$.
1 or γ_5	odd		3/5	3/5	1/5	—
$\gamma_5(1)$	even	symmetrical theory (neutral)	3/7	5/11	3/7	1/7
		neutral theory (symmetrical)	3/7	19/45	3/7	1/7
γ_μ or $\gamma_5\gamma_\mu$	odd		—	3/5	1/5	—
	even		1/7	1/7	3/7	25/63

Fig. 2. Energy dependence of $E^2\sigma_s(E)/\xi_s \cdot s$ for various processes, for $\gamma_A = \gamma_5$ (or 1). The scale of ordinate is arbitrary.



Appendix

I. Calculation of $\sum U_\alpha(p\,; K_\mu, \cdots, K_1)$ and $\sum \bar{U}_\alpha(p\,; K_\mu, \cdots, K_1)$

(A) *Scalar Meson*

For this case $U_\alpha(p\,; K, \cdots, K_1)$ is given by the Eq. (3.13). By using the relations $p^2 + \varkappa^2 = 0$ and $(i\gamma p + \varkappa)u^+(p) = 0$,

$$\left(\frac{i(\gamma\cdot p-K_1)-\varkappa}{(p-K_1)^2+\varkappa^2}\,i\gamma K_1\,u^+(p)\right)_\alpha=\left(\frac{-2(p\cdot K_1)+K_1^2-i\gamma K_1(i\gamma p+\varkappa)}{-2(pK_1)+K_1^2}\,u^+(p)\right)_\alpha=u_\alpha^+(p), \tag{A.1}$$

further

$$\left(\frac{i(\gamma\cdot p-K_1-K_2)-\varkappa}{(p-K_1-K_2)^2+\varkappa^2}\,i\gamma\cdot K_2 u^+(p)\right)_\alpha=\left(\frac{-2(pK_2)+\gamma\cdot K_1+K_2\gamma K_2-i\gamma K_2(i\gamma p+\varkappa)}{-2(p\cdot K_1+K_2)+(K_1+K_2)^2}\,u^+(p)\right)_\alpha$$

$$=\left(\frac{-2(pK_2)+(\gamma\cdot K_1+K_2)\gamma\cdot K_2}{-2(p\cdot K_1+K_2)+(K_1+K_2)^2}u^+(p)\right)_\alpha. \tag{A.2}$$

Adding the expression which is obtained from (A.2) interchanging K_1 with K_2, we have just $u_\alpha^+(P)$. In the same way we finally obtain

$$\sum_{\text{perm}} U_\alpha(p\,;K_n,\cdots,K_1)=u_\alpha^+(p) \tag{A.3}$$

and the corresponding relation

$$\sum \bar{U}_\alpha(p\,;K_n,\cdots,K_1)=(-1)^n\,\bar{u}_\alpha^+(p) \tag{A.4}$$

(B) *Pseudoscalar Meson*

Bringing together γ_5 matrices, Eq. (3.22) becomes as follows:

$$U_\alpha(p\,;K_n,\cdots,K_1)=\left(\gamma_5^n\,\frac{i(\gamma\cdot p-K_1-\cdots-K_n)+(-1)^n\varkappa}{(p-K_1-\cdots-K_n)^2+\varkappa^2}\,i\gamma\cdot K_n\cdots\right.$$

$$\left.\cdots\frac{i(\gamma p-K_1-K_2)-\varkappa}{(p-K_1-K_2)^2+\varkappa^2}\,i\gamma K_2\,\frac{+i(\gamma\cdot p-K_1)+\varkappa}{(p-K_1)^2+\varkappa^2}\,i(\gamma K_1)\,u^+(p)\right)_\alpha. \tag{A.5}$$

When $|\boldsymbol{p}|\gg\varkappa$ and $|\boldsymbol{p}|\gg|\boldsymbol{K}_i|$ $(i=1,2,\cdots,n)$.

$$\left(\frac{+i(\gamma p-K_1)+\varkappa}{(p-K_1)^2+\varkappa^2}\,i\gamma K_1 u^+(p)\right)_\alpha=\left(\frac{2pK_1-K_1^2+i\gamma K_1(i\gamma\cdot p-\varkappa)}{-2(-pK_1+K_1^2)}\,u^+(p)\right)_\alpha$$

$$=\frac{2(pK_1)-K_1^2-\varkappa i\gamma K_1}{-(-2(pK_1)+K_1^2)}\,u^+(p)\sim+u_\alpha^+(p). \tag{A.6}$$

Similarly we can prove

$$\frac{i(\gamma\cdot p-K_1-\cdots-K_n)+(-1)^n\varkappa}{(p-K_1-\cdots-K_n)^2+\varkappa^2}\,i\gamma K_n u^+(p)\doteqdot-\frac{(PK_n)}{(p\cdot K_1+\cdots+K_n)}\,u^+(p) \tag{A.7}$$

so in this assumption

$$u_\alpha(p\,;K_n,\cdots,K_1)\sim\frac{\gamma_5^n(pK_n)}{(pK_1+\cdots+K_n)}\,\frac{(pK_{n-1})}{(p\cdot K_1+\cdots+K_{n-1})}\cdots\frac{(pK_1)}{(pK_1)}\,u_\alpha^+(p). \tag{A.8}$$

By the use of following mathematical relation:

$$\sum_{\text{perm}}\frac{1}{a_1(a_1+a_2)\cdots(a_1+a_2+\cdots+a_n)}=\frac{1}{a_1a_2\cdots a_n}, \tag{A.9}$$

we have

$$\sum_{\text{perm}} U_\alpha(p; K_n, \cdots, K_1) = (\gamma_5{}^n u^+(p))_\alpha \tag{A.10}$$

and

$$\sum_{\text{perm}} \bar{U}_\alpha(p; K_n, \cdots, K_1) = (\bar{u}^+(p)\gamma_5{}^n)_\alpha. \tag{A.11}$$

II. Multiple meson production in the neutral vector meson theory

Here we consider the neutral vector meson theory with vector (f) coupling. Then the process of multiple meson production is analogous to that of photo-bremsstrahlung and gives cross sections much smallar than those in g-coupling. The fundamental wave equation is given by

$$\left\{H(x) + V(x) + \frac{1}{i}\frac{\delta}{\delta C}\right\} \Psi[C] = 0 \tag{A.12}$$

where

$$H(x) = if\bar{\psi}(x)\gamma_\mu\psi(x)\phi_\mu(x) \tag{A.13}$$

and $V(x)$ is the density of the nuclear potential. $\phi_\mu(x)$ is the wave function of the neutral vector mesons and its Fourier decomposition is

$$\phi_\mu(x) = \frac{1}{\sqrt{2}(2\pi)^{3/2}} \sum_{r=1,2,3.} \int \frac{d\boldsymbol{K}_i}{\epsilon_i} v_\mu^r(\boldsymbol{K})(\phi_\mu^*(K)\exp\{-iK\cdot x\} + \phi_\mu(K)\exp\{iK\cdot x\}) \tag{A.14}$$

$v_\mu^r(K)$ is the polarization vector of a vector meson K and satisfies the following relation:

$$\sum_r v_\mu^r(K) v_\nu^r(K) = \delta_{\mu\nu} + \frac{K_\mu K_\nu}{\mu_0^2} \tag{A.15}$$

After the canonical transformation (2.2), we get

$$\langle \bar{\psi}(x)\gamma_A\psi(x)\rangle = \sum_n \left(\frac{1}{\sqrt{2}(2\pi)^{3/2}}\right)^{n+2} f^n \sum_r \sum_{r_0} \int \frac{dp}{E_p} \bar{\phi}_r^+(p) \int \frac{d\boldsymbol{p}_0}{E_{p_0}} \psi(p_0)$$

$$\times \sum_{r_i} \prod_{i=1}^n \int \frac{d\boldsymbol{K}_i}{\epsilon_i} \phi_{r_i}^* \Gamma(p, p_0; K_n, \cdots, K_1) \exp\{i(p_0 - K_1 - \cdots - K_n - p\cdot x)\}. \tag{A.16}$$

This corresponds to Eq. (3.17) and for $E_{p_0} \gg \varkappa, |\boldsymbol{K}_i|$ Γ_A satisfies approximately

$$\sum_{\text{perm}} \Gamma_A(p, p_0; K_n, \cdots, K_1) = \prod_{i=1}^n \left(\frac{(p_0 v^{r_i})}{(p_0 K_i)} - \frac{(p v^{r_i})}{(p\cdot K_i)}\right)(\bar{u}^+(p)\gamma_A u^+(p_0)). \tag{A.17}$$

Then in the expression for the modified nuclear potential $\langle V(x)\rangle$ we have summation as follows:

$$\sum_m \sum_{\text{perm}} \Gamma_A(p, p_0; K_n, \cdots, K_{m+1}) \Gamma_A(q, q_0; K_m, \cdots, K_1)$$

$$\sim \prod_{i=1}^n \left(\frac{(p_0 v^{r_i})}{(p_0\cdot K)} + \frac{(q_0\cdot v^{r_i})}{(q_0\cdot K_i)} - \frac{(p\cdot v^{r_i})}{(p\cdot K_i)} - \frac{(q\cdot v^{r_i})}{(q\cdot K_i)}\right)(p|\gamma_A|p_0)(q|\gamma_A|q_0). \tag{A.18}$$

(A.18) shows that the difference of the nucleon's eigen fields before and after the collision is emitted and the cross section for multiple meson production can be expressed as a product of two independent processes, that is, the nuclear scattering between the nucleons and the shaking off their eigen field. When s neutral mesons are produced,

$$\sigma_s \propto \prod_{i=1}^{s} \int \frac{d\boldsymbol{K}_i}{\epsilon_i} \left(\frac{p_0}{(p_0 \cdot K_i)} + \frac{q_0}{(q_0 K_i)} - \frac{p}{(p \cdot K_i)} - \frac{q}{(q \cdot K_i)} \right)^2 \left(\frac{f^2}{2(2\pi)^3} \right)^s. \quad \text{(A.19)}$$

Where the summation about the polarizations of mesons is already carried out. After integrating over the momentum space, we obtain

$$\sigma_s \propto \frac{1}{s!} \left(\frac{f}{\pi} \log \left(\frac{\sqrt{\mu^2 + \bar{q}^2 \sin^2(\frac{\theta}{2})} + \bar{q} \sin\frac{\theta}{2}}{\mu} \right) \right)^{2s}. \quad \text{(A.20)}$$

$\bar{q}$ is the average momentum per one meson and θ is the angle of deflection of nucleons in the center of gravity system. From (A. 20) the average numer of s when $\bar{q}$ is given becomes,

$$\bar{s} \sim \left[\frac{f}{\pi} \log \left(\frac{\sqrt{\mu^2 + \bar{q}^2 \sin^2\frac{\theta}{2}} + \bar{q} \sin\frac{\theta}{2}}{\mu} \right) \right]^2 \quad \text{(A.21)}$$

which is much smaller than the cases of g-coupling.

References

1) P. Frier and E. P. Ney, Phys. Rev. **77** (1950), 337.
J. J. Lord and Marcel Schein, Phys. Rev. **77** (1950), 337.
2) H. W. Lewis, J. R. Oppenheimer and S. A. Wouthysen, Phys. Rev. **73** (1948), 127.
3) F. Bloch and A. Nordsieck, Phys. Rev. **52** (1934), 54.
4) R. E. Marshak, Phys. Rev. **76** (1949), 1736.
Y. Fujimoto, H. Fukuda, S. Hayakawa and Y. Yamaguchi. Prog. Theor. Phys. **5** (1950), 669.

Progress of Theoretical Physics, Vol. 5, No. 6, November~December, 1950

On the Production of γ-Ray accompanied with the Absorption of π^--Meson by a Proton, I

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(Received June 3, 1950)

Recently, it has been reported* that the π^--meson absorbed in the hydrogen gas produces high energy γ-rays, the spectrum of which has such a form as indicated in Fig. I.**

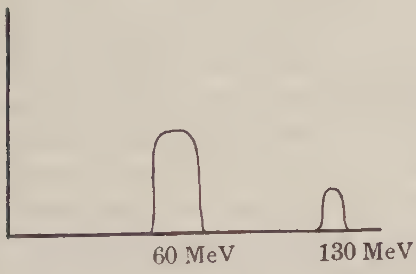

Fig. I

This phenomenon is interpreted to be an elementary process in which a π^--meson is absorbed by a hydrogen nucleus, accompanied with the photon emission. In fact, the Bremsstrahlung of the π^-meson is hardly effective because of its low kinetic energy. The γ-rays associated with the ordinary π-μ decay contribute only to the magnitude of $e^2/\hbar c$ ($=1/137$), and its energy amounts at most to the difference of the rest energies of the π-and μ-meson, that is, 30 Mev.

As seen from Fig I, the spectrum at 130 Mev indicates that one photon, getting almost all the rest energy of a π^--meson, is emitted. On the other hand, a band near 60 Mev suggests that the rest energy of a π^--meson is divided into two photons. The former process is, probably, interpreted to be

$$\pi^- + p \rightarrow N + \gamma.$$

The latter corresponds to

$$\pi^- + p \rightarrow N + 2\gamma,$$

$$\pi^- + p \rightarrow N + \pi^0 \rightarrow N + 2\gamma.$$

Accepting such a interpretation, we can get various informations about the π-meson. In this paper, we shall report a theoretical analysis for the π-meson of scalar (or pseudoscalar) type.

§ 1. Calculations

As the life time that the π^--meson with ordinary kinetic energy ($\sim$10 Mev),

* Letters from Prof. H. Yukawa to Prof. S. Sakata and Dr. Panofsky to Dr. M. Taketani.

** Letter from Prof. H. Yukawa to Mr. S. Hayakawa. We have obtained this information after our preliminary report at Annual Meeting held on April 2, 1950, at Tokyo.

passing through the high pressure hydrogen gas, slows down and is captured into the K-orbit of the hydrogen atom, is considerably short compared with the life time of the ordinary decay[1], we may consider only the probability that a π^--meson in the K-orbit is absorbed by a proton. The processes to be calculated are as follows:

i) A π^--meson is absorbed by a proton, accompanied with one photon emission,

ii) A π^--meson is absorbed by a proton, accompanied with two photons emission,

and

iii) A π^--meson is absorbed by a proton, accompanied with a neutral meson, π^0.

The process iii), if it could occur, demands that the mass of π^0-meson is less than that of π^--meson at least by the mass difference of proton and neutron.

π-and π^0-meson are assumed to be scalar or pseudoscalar-meson, the coupling with the nucleon being taken as to be of scalar-type or ps.-type. The anomalous magnetic moment of the nucleon are included phenomenologically in the interaction. A part of the result of these calculations has already been reported by Marshak and Wightman[2]. We have checked their results and performed more detailed analysis. Though the calculation is similar to theirs, our results obtained are slightly different.* We take the initial state of π^--meson as to be bound in K-orbit. The effect of binding, however, is so small that it hides in the approximations which are taken in the calculations below. Some notes and the results of the calculations are presented.

i) The probability (denoted by $W_{1\gamma}$) that a π^--meson in K-orbit is captured by emitting one photon, is given by

$$W_{1\gamma}(scalar)=8\pi f^2 e^2\left(\frac{1}{\mu}\right)^3 N\left(\frac{\overline{\mu}}{\mu}\right)^2 \mu\left(\frac{\mu}{2M}\right)^2 (\Gamma_N-\Gamma_P)^2,$$

$$\overline{\mu}=\mu(1-\mu/2M), \tag{1}$$

or

$$W_{1\gamma}(pseudoscalar)=8\pi\, f^2 e^2\left(\frac{\mu}{2M}\right)^2\left(\frac{1}{\mu}\right)^3 N\mu(1+\mu/2M)^2,\ \hbar=c=1,$$

where $(\mu/M)^2$ is neglected as compared with 1, μ and M are the mass of π^--meson and nucleon respectively. f is the coupling constant between π^--meson and nucleon. Γ_P and Γ_N are magnetic moment of proton and neutron measured in nuclear magneton. N is the density of π^--meson at the place of nucleon. Noting that π^--meson is in K-orbit, $N=e^6\mu^3/\pi$. These results are coincident with Marshak-Wightman's if we neglect the terms of order (μ/M). In this process, the emitted photon has a sharp maximum at $\mu(1-\mu/2M)$ in the energy spectrum, the breadth of which is some one Mev. Assuming μ to be 278 electron mass, $\mu(1-\mu/2M)$ becomes about 130 Mev.

* Some mistakes seem to be found in their paper.

ii) As the probability (denoted by W_{2T}) that a π^--meson is captured by emitting two photons is expected to be small as compared with W_{1T}, we perform rough approximation in which (μ/M) is neglected with unity and get,

$$W_{2T}(scalar)=\frac{8}{3}f^2e^4\left(\frac{1}{\mu}\right)^3 N\cdot\mu\overline{(\boldsymbol{e}_1,\boldsymbol{e}_2)},$$

or (2)

$$W_{2T}(pseudoscalar)=\frac{4}{5}f^2e^4\left(\frac{1}{\mu}\right)^3\left(\frac{\mu}{2M}\right)^2 N\cdot\mu\cdot\overline{(\boldsymbol{e}_1,\boldsymbol{e}_2)}$$

where $\boldsymbol{e}_1,\boldsymbol{e}_2$ indicate the polarization vectors of the emitted photons. From (2), we can see that two photons polarized perpendicularly to each other, cannot be emitted. $\overline{(\quad)}^2$ means that the value is replaced by its mean value integrated over the angle, so that we can take $\overline{(\boldsymbol{e}_1,\boldsymbol{e}_2)}\sim 1$. Energy spectrum of the γ-rays has such a form as,

$$(1-[1-2k/\mu]^2)dk/\mu, \qquad \text{(scalar)},$$

or

$$(1-[1-2k/\mu]^4)dk/\mu. \qquad \text{(pseudoscalar)}.$$

iii) The probability (denoted by W^n), for one π^0-meson emission, becomes, neglecting the correction of the magnitude $(\mu/M)^2$ and $(\mu/M)(\mu-\mu_0/\mu)$ compared with unity,

$$W_i^n=4\pi f^2\frac{\bar{p}}{\mu}\frac{1}{M+\mu}\frac{\Gamma_i}{(4\mu^2M^2-\mu_0^2)}N,$$

$$\Gamma_1=4\mu^2M(2\mu M\bar{f}-\mu\Delta f)^2,$$

$$\Gamma_2=4M\{2\bar{p}^2M\bar{f}+\mu(4M^2-\mu_0^2)\Delta f\}^2-4\bar{p}^2(2\mu M\bar{f}-\mu_0^2\Delta f)\{2\bar{p}^2M\bar{f}$$
$$+\mu(4M^2-\mu_0^2)\Delta f\}, \tag{3}$$

$$\Gamma_3=\frac{\bar{p}^2}{M}\{(4\mu M^2+2\mu_0^2M-\mu\mu_0^2)\bar{f}\mu M^2\Delta f\}^2,$$

$$\Gamma_4=\bar{p}^2\frac{(2M+\mu)^2}{M}(\mu_0^2\bar{f}-2\mu M\Delta f)^2,$$

$$\bar{p}=\sqrt{([\mu-\Delta M]^2-\mu_0^2)(1+\mu/M)},\quad \Delta f=\frac{1}{2}(f^0-f^0),\quad f=\frac{1}{2}(f_N^0+f_P^0),$$

where $i=1,2,3$ and 4 correspond to the processes $p.s.\ \pi\rightarrow p.s.\ \pi^0$, $s.\pi\rightarrow s.\pi^0$, $p.s.\pi\rightarrow s.\pi^0$ and $s.\pi\rightarrow p.s.\pi^0$ and μ_0 is the mass of π^0-meson. ΔM denotes the mass difference of proton and neutron. f_N^0 and f_P^0 are the coupling constants of π^0-meson with neutron and proton respectively. Δf corresponds to the case in which the

interaction between π^0-meson and nucleon is proportional to the isotopic spin operator τ_3 and $\bar{f}$ to the case in which τ_3 is replaced by unity. $\bar{p}$ is the momentum of the emitted π^0-meson.

§ 2. The spectrum of γ-rays produced by a neutral meson decay

The neutral meson is emitted with the isotropic distribution from a hydrogen nucleus. The momentum and the energy of the neutral meson are denoted by $\bar{p}$ and E_0 respectively. If the neutral meson is of scalar or pseudoscalar type, it decays into two photons, the spectrum of which has some breadth due to Doppler effect. In this section, we shall determine its spectrum. In Fig. II, let x-axis denote the direction of the apparatus. A neutral meson emitted with the angle θ to x-axis decays into two photons, one of which propagates along x-axis. Let $\boldsymbol{k}'_1$, $\boldsymbol{k}'_2$ and $\boldsymbol{k}_1$, $\boldsymbol{k}_2$ be the momentum of the two photons in the rest system of π^0-meson and in the laboratory system respectively. Then we obtain

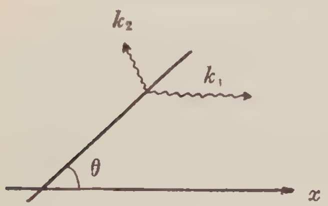

Fig. II

$$(1-\beta \cos\theta)k_1/\sqrt{1-\beta^2}=\mu_0/2, \quad \beta=\bar{p}/E_0. \tag{4}$$

The spectrum of the γ-ray, in the rest-system of π^0-meson, has such a form as $\delta(\mu_0/2-k'_1)dk_1'd\Omega'/4\pi$. Transforming it into the laboratory system, we get

$$\delta\left(\mu_0/2-\frac{1-\beta\cos\theta}{\sqrt{1-\beta^2}}k_1\right)\sqrt{1-\beta^2}/(1-\beta\cos\theta)dk_1d\Omega/4\pi.$$

Integrating over the angle of the emitted π^0-meson, we obtain as the energy spectrum of γ-rays in the laboratory system

$$(\sqrt{1-\beta^2}/\beta)dk/\mu_0. \tag{5}$$

Inserting $\theta=0$ and $\theta=\pi$ in (4), the limits of k are determined as

$$k^{max}=(1+\beta)\mu_0/2\sqrt{1-\beta^2}, \qquad k^{min}=(1-\beta)\mu_0/2\sqrt{1-\beta^2}.$$

So that the γ-ray spectrum has a uniform distribution between k^{max} and k^{min} and vanishes outside the interval. The breadth of the interval is given by

$$k^{max}-k^{min}=2\beta k_0/\sqrt{1-\beta^2}=\bar{p}=\sqrt{([\mu-\Delta M]^2-\mu_0^2)/(1+\mu/M)}$$

the center of which lies at $E_0/2\sim\mu/2$.

If the neutral meson is of vector type, it decays into three photons the energy of which are denoted by k_1', k_2', k_3'. Considering only the statistical weight function to be

$$\infty\, dk_1'dk_2'(k_1'^2+k_2'^2+2k_1'k_2' \cos(k_1'k_2'))^{1/2}k_1'k_2'$$

and taking one photon into account, the spectrum in the rest system of π^0-meson becomes

$$\propto dk' k'^2 (3\mu_0^2 - 6\mu_0 k + 2k^2).$$

Transforming it in the laboratory system with similar method, we obtain

$$\propto \left(\frac{1-\beta\cos\theta}{\sqrt{1-\beta^2}}k\right)^2 \left(3\mu^2_0 - 6\mu_0 \frac{1-\beta\cos\theta}{\sqrt{1-\beta^2}} k + 2\left(\frac{1-\beta\cos\theta}{\sqrt{1-\beta^2}}\right)k^2\right) \frac{\sqrt{1-\beta^2}}{1-\beta\cos\theta} dk\, d\Omega,$$

$$\mu_0/2 \geqq k \frac{1-\beta\cos\theta}{\sqrt{1-\beta^2}} \qquad (6)$$

Integrating over the angle θ with the condition (6), it results

$$\propto \frac{dk}{\mu_0 \bar{p}} \left\{\frac{5}{32}\mu^2_0 - (E_0 - \bar{p})^2\, k^2\left(\frac{3}{2}\mu^2_0 - 2k(E_0 - \bar{p}) + \frac{k^2}{2\mu^2_0}(E_0 - \bar{p})^2\right\},$$

$$k^{max} \geqq k \geqq \frac{\mu_0}{2} \frac{\mu_0}{E_0 + \bar{p}},$$

$$\frac{2k^2 dk}{\mu_0} E_0\{3\mu^2_0 - 2E_0 k(3 + \bar{p}^2/E_0^2) + 2k^2 E_0^2 (1 + \bar{p}/E^2)/\mu_0^2\},$$

$$\frac{\mu_0}{2} \frac{\mu_0}{E_0 + \bar{p}} \geqq k \geqq 0.$$

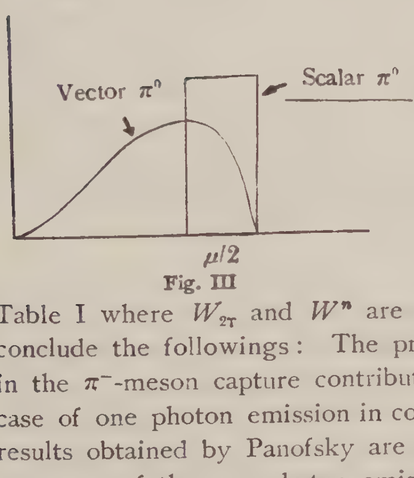

Fig. III

This spectrum differs from that in the case of scalar π_0, in the points that it tails to $k=0$ and has a maximum, while the upper limits of the spectra coincide each other. A qualitative aspect is presented in Fig. III.

§ 3. Numerical results and discussions

The numerical values are presented in Table I where $W_{2\gamma}$ and W^n are compared with $W_{1\gamma}$. From these result we can conclude the followings: The process in which two photons are directly emitted in the π^--meson capture contributes at most 1% as compared with that in the case of one photon emission in consistent with the experiment. The experimental results obtained by Panofsky are definitely due to the competition between the processes of the one photon emission and π^0 emission. From the shape of the spectrum near 60 Mev, the neutral meson of vector-type is excluded. Concerning only to this experiment, of course, the existence of a vector-type neutral meson is not denied, if $\mu - \mu_0 < \Delta M$ or with a scarcely effective value of the coupling constant. The mass of the neutral meson is determined from the breadth of the spectrum near 60 Mev. In fact, Panofsky has found $\mu - \mu_0 < 2.9$ Mev. That means $\mu_0 > 272$ electron mass (assuming μ to be 278 electron mass), while unfortunately,

the upper limit of μ_0 is not so certain. If μ_0 is determined precisely* and the values of the coupling constants (*e.g.* $\bar{f}^2$, Δf^2) are estimated by another experiment, we can obtain the information about the meson type from the relative yield of W_1 and W^n.

In conclusion, the authors should like to express their deep gratitude to Prof. S. Sakata for his valuable discussion and to Dr. M. Taketani and Mr. S. Hayakawa who have kindly told us their informations about the Panofsky's results.

Table I.

W_{1T}(Scalar) $=0.56\times10^{15}\ f^2_s$ /sec; W_{2T}/W_{1T}(Scalar) $=7.1\times10^{-3}$

W_{1T}(Pseudoscalar) $=3.3\times10^{[illegible]}\ f^2_{ps}$/sec; W_{2T}/W_{1T}(Pseudoscalar) $=2.0\times10^{-4}$

W^n/W_{1T}

	μ_0	275.5	275	270	265	260
$p.s.\pi\to p.s.\pi^0$	$f^2\times$	0	12*	38	52	63
	$f\Delta f\times$	0	1.78	5.4	7.2	8.3
	$\Delta f^2\times$	0	0.066	0.19	0.23	0.27
$s.\pi\to s.\pi^0$	$f^2\times$	0	0.047	1.5	3.8	6.7
	$f\Delta f\times$	0	0.088	2.7	7.1	12.7
	$\Delta f^2\times$	0	0.041	1.3	3.3	5.9
$p.s.\pi\to s.\pi^0$	$f^2\times$	0	0.0573	0.02022	0.011	0.029
	$f\Delta f\times$	0	0.061	1.9	4.9	8.8
	$\Delta f^2\times$	0	127	400,	550,	662
$s.\pi\to p.s.\pi^0$	$f^2\times$	0	0.0416	0.0346	0.0211	0.0218
	$f\Delta f\times$	0	0.0343	0.013	0.032	0.055
	$\Delta f^2\times$	0	0.0229	0.091	0.123	0.42
p(Mev)		0	8	26	36	44

* The value means that $W^n/W_{1T}=12\times f^2$ with $\mu=275$ electron mass and in the case that *p.s.* π^--meson is absorbed accompaning *p.s.* π^0-meson emission. Read other values similarly.

References

1) A.S. Wightman, Phys. Rev. **77** (1950), 521.
2) R.E. Marshak and A.S. Wightman, Phys. Rev. **76** (1949), 114.

Note added in proof: We take $\mu-\mu_0<2.9$Mev. Recent experiment, however, indicates $\mu-\mu_0<11m_e$. The main feature of this paper remains unaltered. Some corrections will be reported in the next letter.

* Experimentaly, this will be performed by the comparison of the breadth near 60 Mev with that of the spectrum near 130 Mev which, resulting from the fact that the π^--meson is initialy in K-orbit, corresponds to the interval of about 0.2 Mev.

Progress of Theoretical Physics, Vol. 5, No. 6, November~December, 1950

Gauge-spin Transformations and Wave Equations

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(Received June 5, 1950)

The spin transformation was discussed by E. Schroedinger[1)] and V. Bargmann[2)] as early as 1932. They perceived the abscence of the ground for assuming the spin transformation to be unitary. F. J. Belinfante[3)] and other writers discarded the transformation to preserve the simplicity of the covariant formalism.

We introduce here the spin transformation to set up a linkage between spinor and tensor fields as H. Weyl[4)] did the gauge transformation to connect Dirac's spinor with the electromagnetic potentials.

As we see in the followings, the unitary spin transformation group is so narrow that a linear tensor fields of different ranks having no interaction with one another. The general spin transformation group, on the other hand, requires another spinor field contragradient to ordinary one, but works smooth and clear.

§ 1. Harmonic fields

In the Euclidian space with the metric $ds^2=\sum(dx)^2$ the lower boundary $\mathfrak{r}\varphi^p$ and the upper boundary $\mathfrak{r}^*\varphi^p$ of the antisymmetric tensor field φ^p of the p-th rank are defined as follows

$$\left.\begin{aligned} (\mathfrak{r}\varphi)^{jk\cdots l} &= -\partial_i\varphi^{ijk\cdots l}, \\ (\mathfrak{r}^*\varphi)_{ijk\cdots lm} &= \partial_i\varphi_{jk\cdots lm}-\partial_j\varphi_{ik\cdots lm}+\cdots\pm\partial_m\varphi_{ijk\cdots l}\,. \end{aligned}\right\} \tag{1}$$

When a field has vanishing upper and lower boundaries, i.e.

$$\mathfrak{r}\varphi=0, \qquad \mathfrak{r}^*\varphi=0, \tag{2}$$

the field is called harmonic.[5)]

An example of harmonic field in physics is the electromagnetic field which satisfies the classical Maxwell's equations

$$\partial_i f^{ik}=0, \qquad \partial_i f_{jk}+\partial_j f_{ki}+\partial_k f_{ij}=0.$$

These equations are nothing else than the above-mentioned equation (2). Further all the equations of mesonic fields[6)] are quite of this type, provided one uses the five-dimensional formalism.

E.g. the scalar meson field equations

$$\partial_i\psi=\varkappa\psi_i, \qquad \partial_i\psi^i=\varkappa\psi$$

may be transformed, by imposing $i\psi=\psi^5=\psi_5$, $\partial_5=i\varkappa$, as follows

$$\partial_i\psi_5-\partial_5\psi_i=0, \qquad \partial_i\psi^i+\partial_5\psi^5=0,$$

or

$$\partial_\alpha\psi_\beta-\partial_\beta\psi_\alpha=0, \qquad \partial_\alpha\psi^\alpha=0,$$

where the indices α,β have the range 1, 2, 3, 4, 5.

The equations (2) hold also for vector meson field $\psi_{\alpha\beta}$, pseudovector meson field $\psi_{\alpha\beta\gamma}$, and pseudoscalar meson field $\psi_{\alpha\beta\gamma\delta}$.

Here are no fields of the 0-th and of the fifth rank. If they exist and satisfy (2), they should be stationary.

§ 2. Gauge transformations

To solve the equation (2), we assume

$$\psi^p=\mathfrak{r}^*\varphi^{p-1}$$

so that we may have identically $\mathfrak{r}^*\varphi^p=0$.

The other equation of (2) then will be

$$\mathfrak{r}\varphi^p=\mathfrak{r}\mathfrak{r}^*\varphi^{p-1}=0.$$

If we use a formula

$$-\mathfrak{r}\mathfrak{r}^*-\mathfrak{r}^*\mathfrak{r}=\Delta=\text{Laplacian operator},$$

and impose an auxiliary condition $\mathfrak{r}\varphi^{p-1}=0$ to φ^{p-1}, we have $\Delta\varphi^{p-1}=0$.

There is another way to solve (2). It is to put $\varphi^p=\mathfrak{r}\varphi^{p+1}$ with an auxiliary condition $\mathfrak{r}^*\varphi^{p+1}=0$ and to gain $\Delta\varphi^{p+1}=0$.

If we replace φ^{p-1} by

$$'\varphi^{p-1}=\varphi^{p-1}+\mathfrak{r}^*\varphi^{p-2} \tag{3}$$

where φ^{p-2} is conditioned by the equation $\mathfrak{r}\mathfrak{r}^*\varphi^{p-2}=0$, we have both the same φ^p and the same equation for φ^{p-1} as the fomers respectively.

The transformation (3) is a sort of gauge transformation.

We have also another gauge transformation

$$\varphi^{p-1}=\varphi^{p-1}+\mathfrak{r}\varphi^{p+2}$$

where φ^{p+2} satisfies $\mathfrak{r}^*\mathfrak{r}\varphi^{p+2}=0$.

Let us glance at a special case where p is equal to 2 and the relation $\mathfrak{r}^*\varphi^{II}=0$ always holds. In this case we can put $\varphi^{II}=\mathfrak{r}^*\varphi^{I}$. Under the gauge transformation

$$'\varphi^I=\varphi^I+\mathfrak{r}^*\varphi^0, \qquad ('\varphi_i=\varphi_i+\partial_i\varphi^0). \tag{5}$$

a Pfaff's expression $\varphi_i\,dx^i$ transforms as

$$'\varphi_i\,dx^i=\varphi_i dx^i+\partial_i\varphi_0, \qquad dx^i=\varphi_i dx^i+d\varphi^0.$$

If we assume the invariance of the modified Pfaff's expression $dx^\infty+\varphi_i dx^i$, the gauge transformation (5) is equivalent to a transformation of the coordinate x^∞, $'x^\infty=x^\infty-\varphi^0$.

That the gauge transformation can be reduced to a coordinate transformation is the key to the unified theory initiated by H. Weyl[7] and Th. Kaluza.[8]

The way opened by them, however, does not go to a success when p is greater than 2 and the relation $\mathfrak{r}^*\varphi=0$ does not hold.

When p is 3, the gauge transformation $'\varphi^{II}=\varphi^{II}+\mathfrak{r}^*\varphi^{I}$ $('\varphi_{ij}=\varphi_{ij}+\partial_i\varphi_j-\partial_j\varphi_i)$ contains n independent functions φ_i. Therefore if one wishes to introduce new coordinates, he must deal with n coordinates and will find himself situated in a jungle.

If we modify the harmonic field so that the upper and lower boundaries do not vanish, we have $\mathfrak{r}\varphi^p=\chi^{p-1}$, $\mathfrak{r}^*\varphi^p=\chi^{p+1}$ where χ^{p+1}, χ^{p+1} are certain tensor fields. Putting $\varphi^p=\mathfrak{r}^*\varphi^{p-1}+\mathfrak{r}\varphi^{p+1}$ we can resolve the modified equations into two independent equations $\mathfrak{r}\mathfrak{r}^*\varphi^{p-1}=\chi^{p-1}$, $\mathfrak{r}^*\mathfrak{r}\varphi^{p+1}=\chi^{p+1}$. Examples of modified harmonic fields are found on both hydrodynamics and electrodynamics[9].

§ 3. γ-algebras

We introduce 2^n independent algebras

$$1,\ \gamma^1,\ \gamma^2,\dots\gamma^n,\ \gamma^{12}\ (=-\gamma^{21}),\dots\dots\gamma^{12\cdots n},$$

or

$$\gamma^A\quad (A=0,\ 1,\ 2,\dots,n,\ 12,\dots\dots,12\cdots n)$$

each being antisymmetric in indices and obeying the combination rules

$$\gamma^{Ai}\cdot\gamma^{iB}=\gamma^{AB}\qquad (e.g.\ \gamma^{21}\cdot\gamma^{135}=\gamma^{235},\quad \gamma^1\gamma^1=\gamma^0=1) \tag{6}$$

which enable us to construct γ-algebras from the fundamental algebras $\gamma^1, \gamma^2,\dots\gamma^n$. Further we define a differential operator $D=\gamma^i\partial_i$ and its square, to wit, Laplacian operator $D^2=\Delta=\sum(\partial_i)^2$.

If we make an expression

$$1/p!\cdot\gamma^{ij\cdots k}\ \varphi_{ij\cdots k}=\Phi^p,$$

the equation (2) is equivalent to a single equation $D\Phi^p=0$ since we have

$$\begin{aligned} D\Phi^p&=\gamma^l\partial_l\cdot\ 1/p!\cdot\gamma^{ij\cdots k}\ \varphi_{ij\cdots k}\\ &=1/(p-1)!\cdot\gamma^{j\cdots k}\ \partial^i\varphi_{ij\cdots k}+1/(p+1)!\cdot\gamma^{lij\cdots k}(\partial_l\varphi_{ij\cdots k}\\ &\qquad -\dots\dots\pm\partial_k\varphi_{lij\cdots}).\end{aligned}$$

The representations of the γ^i obtained by E. Cartan[10] and H. Weyl[11] are adequate when n is even, but not so when n is odd. The representation inserted in the "*Classical Groups*"[12] is as follows when n is 2ν

$$\left.\begin{aligned} &\gamma^1=P\times 1\times\dots\times 1,\ \gamma^3=1'\times P\times\dots\times 1,\dots\dots,\ \gamma^{2\nu-1}=1'\times 1'\times\dots\times P,\\ &\gamma^2=Q\times 1\times\dots\times 1,\ \gamma^4=1'\times Q\times\dots\times 1,\ \ \dots,\ \gamma^{2\nu}=1'\times 1'\times\dots\times Q,\end{aligned}\right\} \tag{7}$$

where

$$P=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad Q=\begin{pmatrix}0&i\\i&0\end{pmatrix},\quad 1=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad 1'=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

When $n=2\nu+1$, one adds to (7)

$$\gamma^{2\nu+1}=\gamma^{n}=1'\times 1'\times\ldots\times 1'. \tag{7'}$$

The representations of algebras (7), (7′) are, however, not faithful when $n=2\nu+1$ because there exists a relation

$$\gamma^1\gamma^2\ldots\gamma^n=\gamma^{12\cdots n}=i^{\nu}$$

which violates the combination rules (6).

Therefore we ought to use the following faithful representations

$$\left.\begin{array}{l}\gamma^1=P\times 1\times\ldots\times 1\times 1,\ \gamma^3=1'\times P\times\ldots\times 1\times 1,\quad ,\ \gamma^{2\nu-1}=1'\times 1'\times\ldots\times P\times 1,\\ \gamma^2=Q\times 1\times\ldots\times 1\times 1,\ \gamma^4=1'\times Q\times\ldots\times 1\times 1,\quad ,\ \gamma^{2\nu}=1'\times 1'\times\ldots\times Q\times 1,\end{array}\right\} \tag{8}$$

$$\gamma^{n}=\gamma^{2\nu+1}=1'\times 1'\times\ldots\times 1'\times 1'.$$

We have then

$$\gamma^{12\cdots n}=i^{\nu}(1\times 1\times\ldots\times 1\times 1')\neq i^{\nu}$$

and all the γ^As have matric representations of $2^{\nu+1}$ dimensions,

$$\gamma^A=\begin{pmatrix}\cdots & 0\\ 0 & \cdots\end{pmatrix},$$

in short, each of the representation matrices having $2\times 2^{\nu}$ non-null elements decomposes into a direct sum of two matrices of 2^{ν} dimensions, so that the correspondence between the γ-algebras and the matric algebras of the type (9) is a one-to-one isomorphism.

There exists a matrix β anticommuting with $\gamma^i(i=1, 2, 3,\ldots\ldots, n)$ whose representation is

$$\beta=1'\times 1'\times\ldots\times 1' \qquad n \text{ even},$$

or

$$\beta=1'\times 1'\times\ldots\times 1'\times P \qquad n \text{ odd}.$$

One sees that $i\beta\gamma^i=\beta^i$ $(i=1,2,\ldots,n)$ anticommute with one another and are hermitean.

Spinors having the same number of components as that of the dimensions of matrices are now defined. We have spinors of 4 components when n is equal to 4, ones of 8 components when n is equal to 5.

To obtain certain relations useful for later discussions we form a matrix $F=\gamma^A f_A$ from f_A.

Seeing that

$$tr\ \gamma^A=\begin{cases}1 & A=0,\\ 0 & A\neq 0,\end{cases}\qquad tr\{(\gamma^A)^{\dagger}\gamma_B\}=\begin{cases}1 & (A\to B)\ \text{even transposition},\\ -1 & (A\to B)\ \text{odd transposition},\\ 0 & \text{otherwise},\end{cases} \tag{10}$$

with the trace of matrix defined so as to give $tr\ \mathrm{I}=1$, we have the followings,

$$\left.\begin{aligned} &F=\gamma^A f_A, \quad G=\gamma^A g_A, \\ &tr(F^\dagger G)=\bar{f}^A g_A, \\ &tr(F^\dagger \gamma^i G)=(\bar{f}^i g_0+\ldots)+(\bar{f}^0 g_i+\ldots), \\ &tr(F^\dagger \gamma^A G)=(\bar{f}^A g_0+\ldots)+(\pm\bar{f}^0_i g_A+), \\ &tr\{(DF)^\dagger G\}+tr\{F^\dagger DG\}=\partial_k tr\{F^\dagger\gamma^k G\}. \end{aligned}\right\} \quad \begin{matrix}(11)\\ \\ \\ (12)\\ \\ \end{matrix}$$

If $tr\ (X^\dagger Y)$ vanishes for any X, Y also vanishes.

§ 4. Spinor wave equation

Associating a matrix $X=\beta^i x_i$ to a vector x_i we have the transformation rules of vector and spinor under a reflection A as follows[13)]

$$\left.\begin{aligned} &'X=-AXA \qquad '(\beta X)=A(\beta X)A, \\ &'X_p=(-1)\ AX_pA\ '(\beta^p X_p)=A(\beta^p X_p)A, \\ &'\psi=A\psi, \qquad\qquad '(\psi^\dagger\beta^p X_p\psi)=\psi^\dagger\beta^p X_p\psi. \end{aligned}\right\} \tag{13}$$

Spinor wave equation invariant under reflections should be of the form

$$1/i\cdot\beta^i\partial_i\psi+(\beta f_0+\beta^i f_i+1/2\cdot\beta\beta^i\beta^k f_{ik}+\ldots)\psi=0,$$

or multiplied by β from the left side

$$\gamma^i\partial^i\psi+F\psi=0, \qquad F=f_0+\beta\beta^i f_i+\text{etc.} \tag{14}$$

§ 5. General spin transformations

The condition that the equation (14) should be invariant under general spin transformations

$$\left.\begin{aligned} &'\gamma=S\gamma^k S^{-1}, \qquad S:\ \text{non-singular matrix} \\ &'\psi=S\psi \end{aligned}\right\} \tag{15}$$

determines the transformation rule of F which we assume to be transformed into $'F='\gamma^A\ 'f_A$ by (15).

As the equation (14) transformed by (15).

$$\begin{aligned} '\gamma^k\ \partial_k{}'\psi+'F\,'\psi &= S\gamma^k S^{-1}\partial_k(S\psi)+'FS\psi \\ &= S(\gamma^k\partial_k\psi+\gamma^k\ S^{-1}\ \partial_k\ S\cdot\Psi+S^{-1}{}'FS\psi) \end{aligned}$$

should be of the same form as (14), the transformation rule for F should be

$$S^{-1}\ 'FS+\gamma^k\ S^{-1}\ \partial_k\ S=F$$

or, putting $\quad S^{-1}\ 'FS=\gamma^A{}'f_A=F'$

$$F-F'=\gamma^k\ S^{-1}\ \partial_k\ S$$

i.e.

$$\gamma^A(f_A-'f_A)=\gamma^k\ S^{-1}\ \partial_k S.$$

If we take an infinitesimal transformation

$$\left.\begin{aligned} &S=\exp R=1+R, \\ &R=\gamma^A\ r_A, \quad R:\ \text{infinitesimal}, \end{aligned}\right\} \tag{16}$$

we have

$$F-F'=DR \tag{17}$$

which may be expanded as follows

$$f_0-'f_0=\partial_i r^i,$$
$$f_i-'f_i=\partial_i r_0+\partial_k r^k{}_i,$$
$$f_{ik}-'f_{ik}=\partial_j r_k-\partial_k r_i+\partial_j r^j{}_{ik} \quad \text{etc.}$$

There is no way of eliminating r_A since r_A has the same number of components as f_A has. So we assume here a restriction analogous to the one imposed on the gauge transformation for the electromagnetic potentials and put

$$DDR=\Delta R=0,$$

whence we have a relation

$$DF=DF'$$

and an invariant Lagrangian

$$M=tr\{(DF)^\dagger DF\}.$$

When we assume $F=\gamma^k f_k$, we have

$$DF=\partial_i f^i+\frac{1}{2}\gamma^{ij}(\partial_i f_j-\partial_j f_i)$$

and

$$tr\{(DF)^\dagger DF\}=(\partial_i \bar{f}^i)(\partial_k f^k)+\frac{1}{4}(\partial_i \bar{f}_j-\partial_j \bar{f}_i)(\partial^i f^j-\partial^j f^i).$$

Using a spinor φ which transforms as

$'\varphi=\varphi S^{-1}$ under the spin transformation S

and

$'\varphi=\varphi A^{-1}$ under the reflection A

we obtain the Lagrangian

$$L=\varphi(\gamma^k\partial_k\psi+F\psi)+(\partial_k\psi^\dagger\gamma^k+\psi^\dagger F^\dagger)\varphi^\dagger$$

that gives the wave equation (14) and its conjugate.

Writing $\varphi\gamma^A\psi=\bar{g}^A$, $\gamma^A g_A=G$, we may rewrite L as

$$L=\varphi D\psi+(D\psi)^\dagger\varphi^\dagger+tr(G^\dagger F+F^\dagger G).$$

Therefore adding M to L, we get a Lagrangian $\mathfrak{L}$

$$\mathfrak{L}=L+cM \qquad c: \text{a constant},$$

that gives spinor and tensor wave equations as follows:

spinor equations $\qquad D\psi+F\psi=0, \qquad (D\psi)^\dagger+\varphi^\dagger F^\dagger=0,$

$$\varphi D-\varphi F=0, \qquad D\varphi^{\dagger}-F^{\dagger}\varphi^{\dagger}=0,$$

tensor equations

$$G-c\Delta F=0,$$
$$G^{\dagger}-c\Delta F^{\dagger}=0,$$

where we used the representations of γ adopted in § 3.

Since the action integral

$$I=\int \mathfrak{L}\ dx^1\ dx^2 \ldots dx^n$$

is invariant under spin transformations having 2^n parameters, there may exist 2^n conservation laws. The variation of I due to an infinitesimal spin transformation (16) is substantially equal to

$$\delta I=\int\{\varphi\delta\gamma^k\ \partial_k\psi+tr(G^{\dagger}\ \delta F)+\text{complex conjugate}\}\ dX \tag{18}$$

where $dx=dx^1\ dx^2\ldots dx^n$. Substituting the variations of γ^k and F due to (16)

$$\delta\gamma^k=R\gamma^k-\gamma^k R,$$
$$\delta F=-\gamma^k\partial_k R$$

in (18), we have

$$\begin{aligned}\delta I&=\int\{\varphi(R\gamma^k-\gamma^k R)\ \partial_k\psi-tr\ (G^{\dagger}\ \gamma^k\ \partial_k R)+\text{complex conjugate}\}dX\\ &=\int\{tr(H^{\dagger}R+\partial_k G^{\dagger}\cdot\gamma^k R)+\text{complex conjugate}\}dX\\ &=\int\{tr\{(H+DG)^{\dagger}R+R^{\dagger}(H+DG)\}dX \qquad \text{by (11)}\end{aligned}$$

where $\bar{H}=\gamma^A h_A$, $h^A=\varphi(\gamma^A\gamma^k-\gamma^k\gamma^A)\partial_k\psi$.

Therefore we get conservation laws

$$DG+H=0. \qquad \text{by (12)}$$

§ 6. Unitary spin transformations

To impose the unitary restriction on S makes the situation tedious and complex. We compose from (14) a Lagrangian

$$L=\varepsilon\left\{\frac{1}{2}(\psi^{\dagger}\beta\gamma^i\ \partial_i\ \psi-\partial_i\ \varphi^{\dagger}\cdot\beta\gamma^i\ \psi)+\psi^{\dagger}\beta F\psi\right\},\ \varepsilon;\ \text{ constant n-vector} \tag{19}$$

which gives (14) and

$$-\partial_i\psi^{\dagger}\beta\gamma^i+\psi^{\dagger}\ \beta F=0 \tag{20}$$

(14) and (20) are not consistent unless F has the following property $(\beta F)^{\dagger}=\beta F$, to wit, βF is hermitean. We have therefore

$$F=f_0+i\gamma^i f_i+\frac{1}{2}\gamma^{ij}f_{ij}+\frac{1}{6}\gamma^{ijk}f_{ijk}+\cdots$$

with real components f_A.

The invariance of the Lagrangian under unitary spin transformations gives the transformation rule for F, which turns out to be

$$\frac{1}{2}(\beta\gamma^i S^{-1}\partial_i S^1 . -\partial_i S^{-1}\cdot S\beta\gamma^i)+\beta S^{-1}\,{}'FS=\beta F$$

or

$$F-F'=\frac{1}{2}(\gamma^i S^{-1}\partial_i S-\beta\,\partial_i\,S^{-1}.\;S\beta\gamma^i).$$

If we put $S=\exp iR$, R must be hermitean. When R is infinitesimal,

$$S=1+iR$$

$$F-F'=\frac{i}{2}(\gamma^i\,\partial_i R+\beta\partial_i R\cdot\beta\gamma^i).$$

Putting

$$R=r_0+\gamma^j r_j+\frac{i}{2}\gamma^{jk}\,r_{jk}+\frac{i}{6}\gamma^{jkl}\,r_{jkl}+\dots,\quad r_A \text{ real},$$

we get

$$f_0-{}'f_0=0,$$

$$i\gamma^i(f_i-{}'f_i)=\frac{i}{2}(\gamma^i+\beta\gamma^i)\partial_i r_0,$$

$$\frac{i}{2}\gamma^{ij}(f_{ij}-{}'f_{ij})=\frac{i}{2}(\gamma^i\gamma^j+\beta\gamma^j\,\beta\gamma^i)\partial_i r_j,$$

$$\frac{1}{6}\gamma^{ijk}(f_{ijk}-{}'f_{ijk})=\frac{i}{2}\cdot\frac{i}{2}(\gamma^i\gamma^{jk}+\beta\gamma^{jk}\beta\gamma^i)\,\partial_i r_{jk},\quad \text{etc.}$$

or

$$f_0-{}'f_0=0,$$
$$f_i-{}'f_i=\partial_i\,r_0,$$
$$f_{ij}-{}'f_{ij}=\partial_i\,r_j-\partial_j\,r_i,$$
$$f_{ijk}-{}'f_{ijk}=-\partial_i r_{jk}-\partial_j\,r_{ki}-\partial_k\,r_{ij},\quad \text{etc.}$$

These expressions are nothing else than the gauge transformations (3) discussed in § 2. Eliminating r_A we have

$$\partial_i\,{}'f_0=\partial_i f_0 \qquad,\ \text{say } u_i,$$
$$\partial_i{}'f_j-\partial_j{}'f_i=\partial_i f_j-\partial_j f_i \qquad,\ \text{say } u_{ij},$$
$$\partial_i{}'f_{jk}+\partial_j{}'f_{ki}+\partial_k{}'f_{ij}=\partial_i f_{jk}+\partial_j f_{ki}+\partial_k f_{ij},\ \text{say } u_{ijk},\ \text{etc.}$$

In short, rotations of f_A, say u_A, are invariant under unitary transformations.

So we have invariant Lagrangians

$$M_0=\frac{1}{2}u_i\ u^i,\quad M_1=\frac{1}{4}u_{ij}\ u^{ij},\quad M_2=\frac{1}{6}u_{ijk}\ u^{ijk},\ \text{etc.}$$

Adding these Lagrangians to L, we get a unified Lagrangian

$$\mathfrak{L}=L+c_0M_0+c_1M_1+c_2M_2+\cdots\cdots$$

whence spinor and tensor equations can be derived as follows,

$$D\psi+F\psi=0,$$
$$\varepsilon\psi^\dagger\beta\psi-c_0\Delta f_0=0,$$
$$\varepsilon i\psi^\dagger\beta\gamma\psi-c_1\Delta f^i=0,\quad \text{etc.}$$

As we see in the above calculations, to confine spin transformations within unitary ones is so troublesome and deformed that we cannot have general expressions neat and concise. There exists another choice of Lagrangian. Starting from the modified wave equation

$$\gamma^i\partial_i\psi+iF\psi=0,$$

with hermitean F, and its hermite-conjugate

$$\partial_i\psi^\dagger\cdot\gamma^i-i\psi^\dagger F=0,$$

we have another Lagrangian

$$L=\frac{1}{2i}(\psi^\dagger\ \gamma^i\ \partial_i\ \psi-\partial_i\psi^\dagger\gamma_i\psi)+\psi^\dagger\ F\psi$$

with

$$F=f_0+\gamma^i f_i+\frac{i}{2}\gamma^{ij}f_{ij}+\frac{i}{6}\gamma^{ijk}\ f_{ijk}+\cdots\cdots,\quad f_A\ \text{real.}$$

From the invariance of L under unitary spin transformations, we deduce the transformation rule for F,

$$\frac{1}{2i}(\gamma^i\ S^{-1}\ \partial_iS-\partial_iS^{-1}\cdot\cdot S\gamma^i)+S^{-1}\ 'FS-F=0.$$

Putting $S=1+iR$, we get

$$\frac{1}{2}(\gamma^i\partial_iR+\partial_iR\cdot\gamma^i)+F'-F=0,$$

which may be expanded as follows

$$f_0-'f_0=\partial_ir^i,$$
$$f_i-'f_i=\partial_ir_0,$$
$$f_{ij}-'f_{ij}\overset{\perp}{=}\partial_k\ r^k{}_{ij},$$

$$f_{ijk}-{}'f_{ijk}=\partial_i r_{jk}+\partial_j r_{ki}+\partial_k r_{ij}, \quad \text{etc.}$$

These equations afford invariant tensors

$$\partial_i f_j-\partial_j f_i=u_{ij},$$

$$\partial_i f^{ij}=u^j,$$

$$\partial_i f_{jkl}-\partial_j f_{kli}+\partial_k f_{lij}-\partial_l f_{ijk}=u_{ijkl}, \quad \text{etc.}$$

and invariant Lagrangians $u_{ij}\, u^{ij}$, $u_i u^i$, $u_{ijkl}\, u^{ijkl}$, etc.

This case is too troublesome to treat.

References

1) E. Schroedinger, S-B. Akad. Wiss. Berlin (1932,) 105.
2) V. Bargmann, S-B. Akad. Wiss. Berlin (1932), 346.
3) F. J. Belinfante, Physica **7** (1940), 305.
4) H. Weyl, Z. Phys. **56** (1929), 330.
5) K. Kodaira, Ann. Math. **50** (1949), 587.
6) N. Kemmer, Proc. Roy. Soc. **166** (1938), 127.
7) H. Weyl, *Raum, Zeit und Materie* (1921).
8) Th. Kaluza, S-B. Akad. Wiss. Berlin, (1921), 966.
9) P. A. M. Dirac, Phys, Rev. **74** (1948), 817.
10), 13) E. Cartan, *Theorie des Spineurs*, II (1938).
11), 12) H. Weyl, *Classical Groups* (1946).

Progress of Theoretical Physics, Vol. 5, No. 6, November~December, 1950

On the Decay of a Heavy Dirac Meson.

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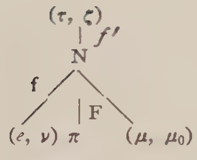
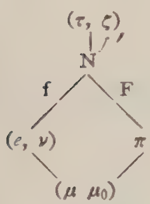
(Received June 14, 1950)

§ I. Introduction



Model II.



Model III.

The various possibilities of the interaction modes between nucleons, π-and μ-mesons and leptons have been discussed by many authors.[1-7] These varieties are critisized by the refined theory,[8] which still contains an inevitable ambiguity. The result depends largely on the prescription to eliminate the divergent integrals. In A the model II, where π-meson decays into μ- and μ_0- mesons or leptons only through a nucleon pair, is only acceptable if the divergent integral is evaluated by the cut off prescription. On the other hand we had to drop the divergent integral by regulator method in order to obtain the life-time for the $\tau \rightarrow 3\pi$ decay.[9] This undesirable inconsistency may suggest that the model II should be given up and one must approve of the model III. To make this conception sure, we want to provide another evidence favourable to the dropping off or regulator prescription.

The process discussed in the model II is the decay of a Boson into Fermions through a nucleon pair. It seems to be more appropriate to discuss the prescription to eliminate the divergent integral in the analogous case like as the transmutation between Boson and Fermion than in the case of that between Boson and Boson. Unfortunately, however, there may not be the further case. Only a case, though suspicious, is the decay of a heavy Dirac meson into a π-meson. We suspected such a possibility analyzing the τ-mesons discovered by Wagner and Cooper.[10] Although the experimental evidences are not yet reliable it is not necessarily meaningless to calculate the life times for the various types of π-meson in order to seek for the prescription evaluating the divergent integral and to analyse the varitorons.

§ 2. Life time of heavy Dirac mesons

The process under considerations is as follows:

$$\tau \rightarrow \pi + \zeta$$

where τ and ζ are Fermions. The mass of τ is put to the observed one ($M = 725\,m_e$), and the mass of ζ is tentatively put equal to that of μ meson or 0 ($m = 210 m_e$ or $m = 0$). The process compared with it is the β-decay of π-meson ($\pi \rightarrow e + \nu$), where the mass of neutrino is 0.

The method of calculation is quite similar to that in A. The life-times for the $\tau \rightarrow \pi + \zeta$ decay are obtained by dividing those for the $\pi \rightarrow e + \nu$ decay by the following factors, the ratio of decay probabilities $\omega_{\tau\pi}/\omega_{\pi e}$.

For scalar π-meson,

$$\frac{\mu}{M}\sqrt{\left\{1-\left(\frac{\mu+m}{M}\right)^2\right\}\left\{1-\left(\frac{\mu-m}{M}\right)^2\right\}}\left\{\left(\frac{M-m}{\mu}\right)^2-1\right\}\frac{f_s'^2}{f_s^2}, \tag{1}$$

where f_s is the scalar type coupling constant between nucleon and Fermions.

For vector π-meson,

$$\frac{\mu}{M}\sqrt{\left\{1-\left(\frac{\mu+m}{M}\right)^2\right\}\left\{1-\left(\frac{\mu-m}{M}\right)^2\right\}}\left\{\left(\frac{M+m}{\mu}\right)^2-1\right\}$$
$$\times\frac{\left\{f_v'+\varepsilon\left(\frac{M-m}{\mu}f_v'\right)\right\}^2+\left\{\left(\frac{M-m}{\mu}\right)f_v'-\varepsilon f_t'\right\}^2}{2f_v^2+\varepsilon^2 f_t^2}, \tag{2}$$

where f_v and f_t are the vector and tensor type coupling constants between nucleon and Fermions respectively, and ε is given by

$$\varepsilon = \frac{\left(\frac{2\varkappa}{\mu}A\right)\cdot F+2\left(\left(\frac{\varkappa}{\mu}\right)^2 A+B\right)\cdot G}{(2B)\cdot F+\left(\frac{\varkappa}{\mu}A\right)\cdot G}.$$

where F and G denote the coupling constants of vector and tensor couplings of nucleon and π-meson respectively, and A and B are the following expressions containing a divergent integrals:

$$A = 2\log\infty+\frac{1}{6}\left(\frac{\mu}{\varkappa}\right)^2+\frac{1}{60}\left(\frac{\mu}{\varkappa}\right)^4+\cdots\cdots,$$

$$B = \frac{1}{3}\log\infty+\frac{1}{30}\left(\frac{\mu}{\varkappa}\right)^2+\frac{1}{280}\left(\frac{\mu}{\varkappa}\right)^4+\cdots\cdots.$$

For pseudovector π-meson,

$$\frac{\mu}{M}\sqrt{\left\{1-\left(\frac{\mu+m}{M}\right)^2\right\}\left\{1-\left(\frac{\mu-m}{M}\right)^2\right\}}\left\{\left(\frac{M-m}{\mu}\right)^2-1\right\}$$
$$\times\frac{\left(1+2\left(\frac{M+m}{\mu}\right)^2\right)(Ff'_{pv})^2+\left(\left(\frac{M+m}{\mu}\right)^2+2\right)(Gf_t')^2}{(Ff'_{pv})^2+2(Gf_t)^2}, \tag{3}$$

where f_t and f_{pv} are tensor and pseudovector couplining constants of Fermions, and F and G represent pseudovector and tensor couplings of π-mesons.

For pseudoscalar π-meson,

$$\frac{\mu}{M}\sqrt{\left\{1-\left(\frac{\mu+m}{M}\right)^2\right\}\left\{1-\left(\frac{\mu-m}{M}\right)^2\right\}}\left\{\left(\frac{M+m}{\mu}\right)^2-1\right\}\alpha^2\frac{f'^2}{f^2}, \tag{4}$$

where $\alpha=1$ or $((M+m)/m_e)^2$ according to the Fermi coupling being pseudoscalar or pseudovector. In the mixed case

$$\alpha=\frac{f'_{ps}\left\{F\left(-\left(\frac{x}{\mu}\right)^2A+3B\right)+G\left(\frac{x}{\mu}A\right)\right\}+f'_{pv}\left(\frac{M+m}{\mu}\right)A\left(-\left(\frac{x}{\mu}\right)F+2\left(\frac{x}{\mu}\right)^2G\right)}{f_{ps}\left\{\qquad ,, \qquad\right\}+f_{pv}\left(\frac{m_e}{\mu}\right)A\left(\qquad ,, \qquad\right)}.$$

In the above expression the mass of nucleon is represented by x and the coupling constants between nucleon and τ, ζ or e, ν is denoted by f' or f, respectively. We may tentatively adopt the relation $(f'^2/f^2)\sim 10^8$, considering the small abundance of such τ mesons.

In the expression A and B, there occur the divergent integrals

$$\log\infty=\int_0^\infty\frac{k^2dk}{\sqrt{k^2+x^2}^3},$$

The results are different whether we prescribe this integral by the regulator, the simple drop off or the cut off. Accounting for the larger value of f'^2 we see the cut off prescription gives too short a life time to explain the track length of several hundred microns. The cut off prescription would not be applicable if the nature of the τ meson with the mass $625m_e$ were established as above. We can not say which of the remaining two prescriptions is preferable, because of the large ambiguity in the magnitudes of coupling constants and in the type of couplings, though the regulator method gives more plausible result. Here we show the life times of τ-meson only in the case of the regulator prescription in the following Table.

Table. Life time of τ-meson.

Type of π-meson	Coupling of π with nucleon	Coupling of τ with nucleon	Life time in sec. $m=210\ m_e$	Life time in sec. $m=0$
scalar	s	s	$5.7.\ 10^{-10}$	$5.0.10^{-9}$
	s	v	forbidden.	
	v	s	forbidden	
	v	v	forbidden	
vector	v	v	$3.4.10^{-10}$	$9.8.10^{-10}$
	v	t	$4.9.10^{-13}$	$1.2.10^{-12}$
	t	v	$1.1.10^{-12}$	$3.4.10^{-12}$
	t	t	$1.1.10^{-10}$	$2.6.10^{-10}$
pseudo-vector	pv	pv	$7.9.10^{-10}$	$1.2.10^{-9}$
	pv	t	forbidden	
	t	pv	forbideen	
	t	t	$5.8.10^{-8}$	$8.0.10^{-9}$
pseudo-scalar	ps	ps	$1.4.10^{-8}$	$1.8.10^{-8}$
	ps	pv	$1.4.10^{-10}$	$2.3.10^{-13}$
	pv	ps	$8.3.10^{-12}$	$1.0.10^{-11}$
	pv	pv	$1.6.10^{-10}$	$2.6\ 10^{-10}$

References

1) M. Taketani, S. Nakamura, K. Ono and M. Sasaki, Phys. Rev. **76** (1949), 60.
2) S. Sasaki, S. Ōneda and S. Ozaki, Sci. Rep. Tohoku Univ. **33** (1949), 77.
3) H. Yukawa, Rev. Mod. Phys. **21** (1949), 474.
4) L. I. Schiff, Phys. Rev. **76** (1949), 303, 1266.
5) M. Taketani, S. Nakamura and M. Sasaki, Phys. Rev. **76** (1949), 852: Prog. Theor. Phys. **4** (1949), 552.
6) J. Steinberger, Phys. Rev. **76** (1949), 1186.
7) M. Ruderman and R. Finkelstein, Phys. Rev. **76** (1949), 1458.
8) H. Fukuda, S. Nakamura, M. Sasaki, M. Taketani and K. Ono, Prog Theor. Phys. **5** (1950), 740.
9) H. Fukuda, S. Hayakawa and Y. Miyamoto, Prog. Theor. Phys. in press cited as B.
10) S. Hayakawa and Y. Yamaguchi, Prog. Theor. Phys. **4** (1949), 579.
11) P. H. Fowler, Phil. Mag. **41** (1950), 169.
12) H. W. Lewis, J. R. Oppenheimer and S. A. Wouthuysen, Phys. Rev. **73** (1948), 127.

Progress of Theoretical Physics, Vol. 5, No. 6, November~December, 1950

Some Remarks on the Condensation Phenomena.

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(Received June 19, 1950)

§ 1. Introduction

The fact which characterizes the condensation phenomena of the imperfect gases may be the following two points.

(A) Isothermal compression of gas produces another phase, whose physical properties are much different from those of gas, i.e., liquid phase; and further compression makes at last the whole system liquid phase only.

(B) In some range of the above mentioned process, the pressure remains constant against the change of the volume, i.e., the pv-curve has a horizontal part.

(A) and (B) are at first sight independent from each other. The theory presented heretofore seems to aim to prove principally (B) (for example, works of Mayer[1], Kahn and Uhlenbeck[2], Born and Fuchs[3], etc.). The fact (A), however, seems to be more fundamental, so that the present paper aims to prove (A) as well as (B), and to investigate the correspondence between the "coexistence state of the two phases" in (A) and the "horizontal part of the pv-curve" in (B).

§ 2. Partition function

Let us consider the system composed of N molecules contained in the vessel whose volume is V. We divide V in equal cells, whose number and volume are denoted by m and τ respectively, i.e., $\tau = V/m$.

The number of cells each of which contains i molecules is denoted by m_i. Then

$$\sum_{i=0}^{N} m_i = m, \qquad \sum_{i=0}^{N} i m_i = N. \tag{1}$$

The following two facts are to be noted.

(i) If m_i has two sharp maxima against i, the evidence of the coexistence of two phases may be shown.

(ii) When we regard each cell as a system and use the suitably approximate partition function for each cell, we can reconstruct the partition function of the whole system.

The partition function of the whole system Q is

$$Q_N=\frac{1}{N!\,h^{3N}}\int\cdots\int\exp\left(-\frac{H(p,q)}{kT}\right)dp_{1x}\ldots dz_N$$

$$=\left(\frac{2\pi mkT}{h^2}\right)^{\frac{3N}{2}}\Omega(T,V,N), \tag{2}$$

where

$$\Omega(T,V,N)=\frac{1}{N!}\int\cdots\int\exp\left(-\frac{U(q)}{kT}\right)dx_1\ldots dz_N. \tag{3}$$

Here the Hamiltonian and the potential energy of the total system are denoted by $H(p,q)$ and $U(q)$ respectively.

The partition function of each cell (denoted as Q^*) which contains i molecules is

$$Q_i^*=\left(\frac{2\pi mkT}{h^2}\right)^{\frac{3i}{2}}\Omega^*(T,\tau,i), \tag{4}$$

where

$$\Omega^*(T,\tau,i)=\frac{1}{i!}\int\cdots\int\exp\left(-\frac{U(x_1,\ldots z_i)}{kT}\right)dx_1\ldots dz_i$$

$$=\frac{\tau^i}{i\,!}\exp\left(-\frac{U_i}{kT}\right), \tag{5}$$

where U_i is promised to be defined by (5).

Now we assume: the interaction energy between two particles which belong to different cells can be neglected. Then, the integration of (2) in $3N$-dimensional space is replaced by the summation (with regard to the set of m_i) of the product of the integration in $3i$-dimensional cell (whose volume is τ^i), corresponding to the division of the 3-dimensional space V. Hence

$$\Omega(T,V,N)=\frac{1}{N!}\sum^{\alpha}\frac{m_0!}{m_0!\,m_1!\ldots m_N!}\,\frac{N!}{(0!)^{m_0}(1!)^{m_1}\ldots(N!)^{m_N}}$$

$$\cdot 1^{m_0}\cdot\left[\int\exp(-U(q_1)/kT)dx_1\,dy_1\,dz_1\right]^{m_1}\cdot$$

$$\cdot\left[\int\exp(-U(q_1,q_2)/kT)\cdot dx_1\ldots dz_2\right]^{m_2}\cdots\cdots$$

$$\cdots\left[\int\exp(-U(q_1\cdots q_N)/kT)\,dx_1\ldots dz_N\right]^{m_N}$$

$$=m!\sum^{\alpha}\prod_{i=0}^{N}\frac{\{\Omega^*(T,\tau,i)\}^{m_i}}{m_i\,!}, \tag{6}^*$$

where $\sum^{\alpha}$ means the summation with regard to all sets of m_i under the restriction

*The similar expression is reported to be derived by Van Hove.[4]

of (1). Therefore, we can construct Ω_N from Ω_i^* by (6). Now we shall show that Ω_N reconstructed from approximate Ω_i^* gives more improved approximation.

We replace Ω_N by the greatest term in $\sum^{a}$. It means to pick up the most probable microcanonical ensemble. The results obtained in such a way can be proved to be true in the case $N\to\infty$. (See Appendix). The set of m_i which maximizes the term in $\sum^{a}$ is obtained by the method of Lagrangian multiplier.

$$\delta\sum_{i=0}^{N}\log\frac{(\Omega_i^*)^{m_i}}{m_i!}=0, \tag{7}$$

$$\delta\sum_{i=0}^{N}m_i=0, \qquad \delta\sum_{i=0}^{N}im_i=N. \tag{8}$$

Hence

$$m_i=A\Omega_i^*\, z^i=Ax_i\,\xi^i \tag{9}$$

$$=A\frac{\xi^i\exp(-U_i/kT)}{i!}, \tag{10}$$

where

$$x_i=\frac{\exp(-U_i/kT)}{i!},\qquad \xi=z\tau. \tag{11}$$

The undetermined multipliers A and z (or ξ) are determined from

$$\left.\begin{aligned} m&=A\sum_{i=0}^{N}\Omega_i^* z_i=A\sum^{N} x_i\,\xi^i,\\ N&=A\sum_{i=0}^{N} i\Omega_i^* z^i=A\sum_{i=0}^{N} ix_i\,\xi^i \end{aligned}\right\} \tag{12}$$

Substituting (10) into (6), we get

$$\log\Omega(T,V,N)=-N\log z+m\,\log\sum^{N}\Omega_i^* z^i$$

$$=-N\log(\xi/\tau)+m\log\sum_{i=0}^{N}x_i\,\xi^i. \tag{13}$$

Let us change the number of molecules N for given constant volume V and constant number of cells m, instead of changing the total volume V under given N. The pressure p is

$$\frac{p}{kT}=\frac{d\log\Omega}{d(V/N)}$$

$$=-\frac{N^2}{V}\frac{d}{dN}\left(-\log\frac{\xi}{\tau}+\frac{m}{N}\log\sum x_i\xi^i\right)$$

$$=\frac{m}{V}\log\sum x_i\xi^i$$

$$=\frac{m}{V}\log\sum \Omega_i^* z^i. \tag{14}$$

The specific volume v is, from (12),

$$v=\frac{V}{N}=\frac{m\tau}{N}=\tau\frac{\sum x_i\xi^i}{\sum ix_i\xi^i},$$

or (15)

$$\bar{i}=\frac{N}{m}=\frac{\sum ix_i\xi^i}{\sum x_i\xi^i}=\frac{\sum i\Omega_i^* z^i}{\sum \Omega_i^* z^i}$$

§ 3. Existence of the condensed phase

Here we shall give the general proof of the existence of the condensed phase. We put

$$G(i)=\frac{d}{di}\log m_i=\log\xi-\frac{1}{kT}\frac{dU_i}{di}-\log i. \tag{16}$$

The value of i which maximizes m_i is given by the roots of

$$G(i)=0. \tag{17}$$

As we have already mentioned, the existence of the two sharp maxima of m_i shows the coexistence of the two phases. To study $G(i)$, we shall estimate U_i.

At first we approximate (5) by the following way. By the mean value theorem U_i is the potential energy of the appropriate configuration (cf. (5)). We replace U_i by $i\,\epsilon_i$, which is the potential energy of the mean distance configuration. The molecular force of the imperfect gas is attractive at large distance, and repulsive at short distance; that is, the potential between two molecules $\phi(r)$ (r is the distance between two molecules) has generally the following nature:

$\phi(r)\to\infty$, when $r\to 0$,

$\phi(r)$ has a negative minimum at some positive r,

$\phi(r)\to -0$, when $r\to\infty$.

Then we may expect that:

$\epsilon_i\to 0$, when $i\to 0$ (infinite dilution),

ϵ_i has a negative minimum at some value of i,

$\epsilon_i\to\infty$, when $i\to\infty$ (molecules are very close together).

In fact, we can calculate ϵ_i by using the Lennard-Jones potential, i.e.,

$$\epsilon_i = \frac{1}{2} \sum_{n=1}^{\infty} \phi(n(\tau/i)^{\frac{1}{3}}) 4\pi n^2.$$

Fig. 1 shows the behaviour of $\phi(r)$, ϵ_i, and dU_i/di.

Next, we adopt the hole theory for calculating U_i of each cell. For example[5], the free energy is given by

$$F = -\frac{zN\chi}{2} + \frac{z}{2} kT \left\{ N \log \frac{\beta - 1 + 2\theta}{\theta(\beta+1)} + (L-N) \log \frac{\beta + 1 - 2\theta}{(1-\theta)(\beta+1)} \right\}$$

$$+ kT \left\{ N \log \theta + (L-N) \log(1-\theta) \right\} - NkT \log (V/L),$$

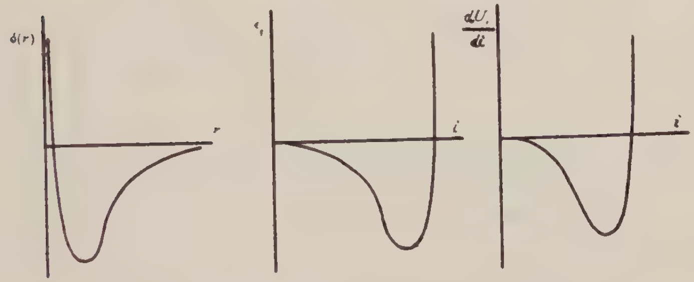


Fig. 1. The behaviour of $\phi(r)$, $i\epsilon_i$, and dU_i/di.

where

$$\theta = N/L, \quad B = \{1 + 4N(L-N)(e^{\chi/kT} - 1)/L^2\}^{\frac{1}{2}},$$

L is the number of sites, χ is the energy per one pair of molecules, and z is the number of the nearest neighbours. From this expression of the free energy, we can know the behaviour of U_i. Considering the roughness of the approximation near $N \to 0$ and $N \to L$, we can regard Fig. 1 as the general behaviour of U_i under the constant temperature. That the behaviour of dU_i/di is similar to that of U_i can be seen by the graphical differentiation.

We can see immediately the behaviour of $G(i)$ graphically. Fig. 2 (a) shows the case when the temperature is sufficiently low, and Fig. 2 (b) the case it is sufficiently high. The incerease of ξ means the upward shift of the whole curve by the amount $\log \xi$.

As Fig. 2(a) shows, in a certain range of ξ ($\xi_1 < \xi < \xi_2$), $G(i)$ has three roots, which are denoted from the smaller one by i_g, i'. and i_l. m_i takes maximum, minimum, and maximum respectively. Fig. 3 shows this behaviour of m_i. This region of ξ is that of the coexistence of the two phases in the meaning of (A).

For any ξ outside of this region, $G(i)$ has only one root and m_i has only one maximum. When $\xi < \xi_1$, it corresponds to the gaseous phase, and when $\xi > \xi_2$, it corresponds to the liquid phase. (The fugacity ξ/τ is monotonic increasing with N/m).

If the temperature is sufficiently high, m_i has only one maximum for any value of ξ. This means the temperature above the critical point.

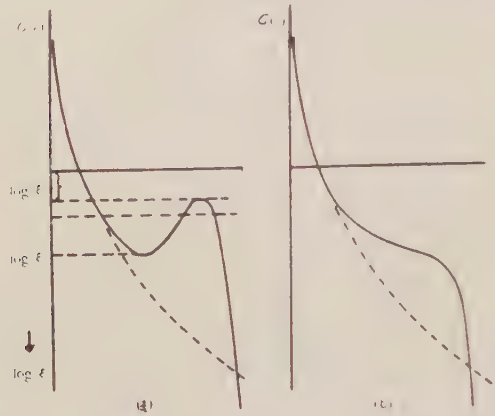


Fig. 2 The relation between $G(i)$ versus i.
(a) The case where $T<T_c$.
(b) The case where $T<T_c$.

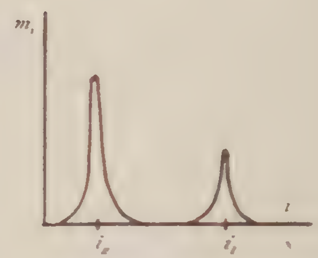


Fig. 3. m_i

The critical temperature T_c can be obtained by eliminating i from

$$G'(i)=0 \text{ and } G''(i)=0. \tag{18}$$

When the intermolecular force is repulsive, there are no condensation phenomena, since ϵ_i is monotonic increasing and $G(i)$ is monotonic decreasing.

§ 4. Equation of state

We can get the equation of state by eliminating ξ from (14) and (15). In evaluating $\sum x_i\xi^i$, we replace $\sum$ by $\int\cdots di$. Remarking $m_i=Ax_i\xi^i$ has one or two sharp maxima, we approximate the integrand by two Gaussian function. Then

$$\sum x_i\xi^i=\int\frac{m_i}{A}\,di=\frac{1}{A}\int\exp\log m_i\;di$$

$$=\frac{m_{ig}}{A}\int\exp\left[\frac{1}{2}\left(\frac{d}{di}G(i)\right)_{i_g}(i-i_g)^2\right]di$$

$$+\frac{m_{il}}{A}\int\exp\left[\frac{1}{2}\left(\frac{d}{di}G(i)\right)_{i_l}(i-i_l)^2\right]di$$

$$=\gamma[i_g(\xi)]+\gamma[i_l(\xi)]. \tag{19}$$

And

$$\sum ix_i\xi^i=i_g\gamma[i_g(\xi)]+i_l\gamma[i_l(\xi)]. \tag{20}$$

where

$$\gamma[i(\xi)]=\exp\{a[i(\xi)]\}/\beta[i(\xi)],$$
$$a[i(\xi)]=\left(i-\frac{U_i}{kT}+\frac{i}{kT}\frac{dU_i}{di}\right), \tag{21}$$
$$\beta[i(\xi)]=\left(1+\frac{i}{kT}\frac{d^2U_i}{di^2}\right)^{\frac{1}{2}}$$

Hence

$$p\tau/kT=\gamma[i_g(\xi)]+\gamma[i_l(\xi)], \tag{22}$$
$$v=\tau\frac{\gamma[i_g(\xi)]+\gamma[i_l(\xi)]}{i_g\,\gamma[i_g(\xi)]+i_l\,\gamma[i_l(\xi)]}. \tag{23}$$

The meaning of these equations is: i_g and i_l are determined from $G(i)=0$ under given ξ; $a[i_g(\xi)]$, $\beta[i_g(\xi)]$, $\gamma[i_g(\xi)]$ and so on are determined from them, and p and v are determined from these values; that is, all these quantities are functions of ξ.

If $\xi<\xi_1$, or $\xi>\xi_2$, each equation (19) (20) (22) (23) is promised to take only the first or the second term respectively.

At first, we consider the case of a single phase in the sense (A). If $\xi<\xi_1$, then

$$v=\tau/i_g(\xi), \tag{24}$$
$$\begin{aligned}p\tau/kT&=\log\gamma[i_g(\xi)]=a[i_g(\xi)]-\log\beta[i_g(\xi)]\\&=i_g-\frac{U_{i_g}}{kT}+\frac{i_g}{kT}\left(\frac{dU_i}{di}\right)_{i_g}-\frac{1}{2}\log\left[1+\frac{i_g}{kT}\left(\frac{d^2U_i}{di^2}\right)_{i_g}\right].\end{aligned} \tag{25}$$

The last term can be neglected.

If $\xi>\xi_2$, i_g in (24) and (25) must be replaced by i_l.

Next, we consider the case where $\xi_1<\xi<\xi_2$ holds. From the fact that $\beta[i(\xi)]$ is the slowly varying function of i or ξ, and $\exp[a\{i(\xi)\}]$ is the exponential function of i, the behaviours of (22) and (23) are both determined mainly by the behaviour of $a[i(\xi)]$; that is,

$$\text{if } a[i_g(\xi)]>a[i_l(\xi)], \text{ then } a[i_g(\xi)]\gg\gamma[i_l(\xi)],$$
$$\text{if } a[i_g(\xi)]<a[i_l(\xi)], \text{ then } \gamma[i_g(\xi)]\ll\gamma[i_l(\xi)].$$

We represent the value of ξ which makes

$$a[i_g(\xi)]=a[i_l(\xi)] \tag{26}$$

ξ_{COND}. Then (24) and (25) hold even if in the case of $\xi_1<\xi<\xi_{COND}$, while in the case of $\xi_{COND}<\xi<\xi_2$, i_g is replaced by i_l.

Now, in the case $\xi\simeq\xi_{COND}$, that is, in the range where the sign of the difference $a[i_g(\xi)]-a[i_l(\xi)]$ changes, τ/v decreases abruptly from $i_g(\xi_{COND})$ to $i_l(\xi_{COND})$. On the other hand, p/kT at $\xi\simeq\xi_{COND}$ changes continuously (but its deri-

vative is discontinuous). The feature of $v(\xi)$ is shown in Fig. 4 (a). Therefore, the pv-diagram has necessarily the horizontal part as shown in Fig. 5.

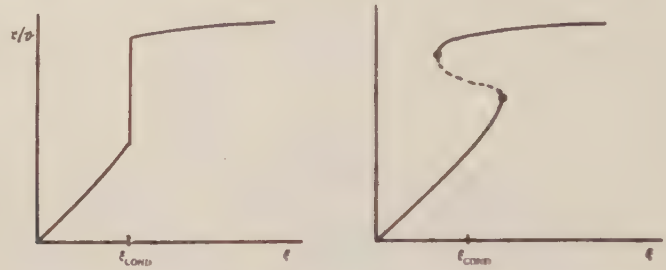


Fig. 4. (a) ξ versus τ/v, (b) ξ versus i_g i^l and i_l .
When there exists single phase only, τ/v is equal to i_g or i_l .

§ 5. Physical consideration on the coexistence of the two phases

Let us investigate the feature of the condensation in further detail. The maximum of m_t is at first only at i_n. In increasing ξ, $i_g(\xi)$ becomes larger, and at $\xi=\xi_1$, i_l appears. In the range $\xi_1<\xi<\xi_{COND}$, both $i_g(\xi)$ and $i_l(\xi)$ increase with ξ. At the neighbourhood of ξ_{COND} both $i_g(\xi)$ and $i_l(\xi)$ remain constant, but $m_{i_g}(\xi\cong\xi_{COND})$ decreases and $m_{i_l}(\xi\cong\xi_{COND})$ increases. In the range $\xi_{COND}<\xi<\xi_2$, both $i_g(\xi)$ and $i_l(\xi)$ increase. At $\xi=\xi_2$, i_g, disappears.

The horizontal part of the pv-curve, i.e., the condensation in the sense (B) arises at $\xi=\xi_{COND}$, on the other hand, two phases in the sense (A) exist between ξ_1 and ξ_2.

We can explain the supersaturated state as follows. Let us imagine that in the range $\xi_1<\xi<\xi_{COND}$ only the gaseous phase is forced to exist by some inhibition, then there is no contribution of i_l in (22) and (23), and pv-curve is extended to ξ_2 without showing any irregularity at $\xi=\xi_{COND}$; this is the phenomena of supersaturation. The phenomena of superexpansion of liquids are explained similarly. These features are shown by the dotted curve in Fig. 5.

It is sure that always $dp/dv\leqq 0$, and the part where $dp/dv>0$ does not exist. This part is usually excluded by the consideration that it contradicts with the real phenomena. While our treatment proves it theoretically.

The physical quantity mentioned in § 1 (A) which distinguishes the gaseous and liquid phase essentially is the density. That is, there is a distinction of gas and liquid only under the coexistence of the two parts with different densities.

Our treatment shows the additivity of the thermodynamical extensive variables. We consider free energy for example. We denote the free energy of the total system, that of one molecule, number of particles, and specific volume at the starting point of the condensation (in the sense (B)), F', f, N', and v' respetively; then

$$F'=F'_{kin}+N'\log(\xi/\tau)-m\log\sum x_i\xi^i.$$

Hence

$$f' = f'_{kin} + \log(\xi/\tau) - (m/N')\log\sum x_i \xi^i$$
$$= f'_{kin} + \log(\xi/\tau) - pv', \tag{27}$$

where F_{kin} and f_{kin} are the terms which arise from the translational partition function.

Similarly

$$f'' = f''_{kin} + \log(\xi/\tau) - pv'', \tag{28}$$

where the double prime means the value at the end of condensation in the sense (B).

Now under the coexistence state of two phases (in the sense (B)), we denote the number of particles of gaseous phase, that of liquid phase and the total number by N_g, N_l and N; and ξ may be regarded as constant (ξ_{COND}), then

$$F(N_g, N_l) = F_{kin} + N\log(\xi/\tau) - m\log\sum x_i \xi^i$$
$$= (N_g + N_l) f_{kin} + (N_g + N_l)\log(\xi/\tau) - pV$$
$$= N_g(f_{kin} + \log(\xi/\tau) - pv') + N_l(f_{kin} + \log(\xi/\tau) - pv'')$$
$$= N_g f' + N_l f''. \tag{29}$$

This is the additivity of free energy. The additivity of entropy, energy, etc. will be clear.

We notice that our treatment does not show the existence of the mixing free energy, while our designation of configuration is similar to the gas mixture.

Moreover, the equation of Clapeyron-Clausius can be derived. From (14), the pressure at the condensation point p_{COND} is given by

$$p_{COND} = \frac{m}{V} kT \log\sum \Omega_i^* z^i_{COND},$$

$$\frac{dp_{COND}}{dT} = \frac{m}{V}\, k \log\sum \Omega_i^* z^i_{COND} + \frac{m}{V}\, kT \frac{\partial \log\sum \Omega_i^* z^i_{COND}}{\partial T}. \tag{30}$$

On the other hand, the entropy S is

$$S = k\log Q + kT\frac{\partial \log Q}{\partial T}$$
$$= S_{kin} - kN\log z + mk\sum \Omega_i^* z^i$$
$$-kNT\frac{\partial \log z}{\partial T} + mkT\frac{\partial \log\sum \Omega_i^* z^i}{\partial T}. \tag{31}$$

Denoting the entropy per molecule at the starting point and the end point of condensation (in the sense (B)) by s' and s'' respectively, their difference is,

$$s' - s'' = \frac{S'}{N'} - \frac{S''}{N''}$$

$$=\frac{m}{N'}k\log\sum\Omega_i^*z^i{}_{COND}-\frac{m}{N''}k\log\sum\Omega_i^*z^i{}_{COND}$$

$$+\frac{m}{N'}kT\frac{\partial\log\sum\Omega_i^*z^i{}_{COND}}{\partial T}-\frac{m}{N''}kT\frac{\partial\log\sum\Omega_i^*z^i{}_{COND}}{\partial T}$$

$$=\left(\frac{m}{N'}-\frac{m}{N''}\right)\left(k\log\sum\Omega_i^*z^i{}_{COND}+kT\frac{\partial\log\sum\Omega_i^*z^i{}_{COND}}{\partial T}\right). \tag{32}$$

Therefore

$$\frac{dp_{COND}}{dT}=\frac{m}{V}\frac{s'-s''}{\left(\frac{m}{N'}-\frac{m}{N''}\right)}=\frac{s'-s''}{v'-v''}. \tag{33}$$

Here the proof is completed.

Using the value of " ϵ_i " ins‘ ɹ of U_i, we can calculate the pv-curve of argon (Fig. 5).

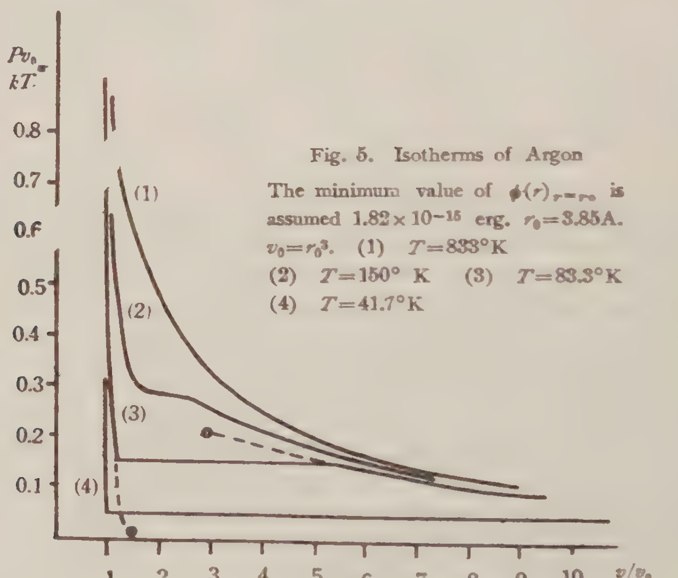


Fig. 5. Isotherms of Argon
The minimum value of $\phi(r)_{r=r_0}$ is assumed 1.82×10^{-15} erg. $r_0=3.85$A. $v_0=r_0^3$. (1) $T=833°$K (2) $T=150°$ K (3) $T=83.3°$K (4) $T=41.7°$K

§ 6. Discussions

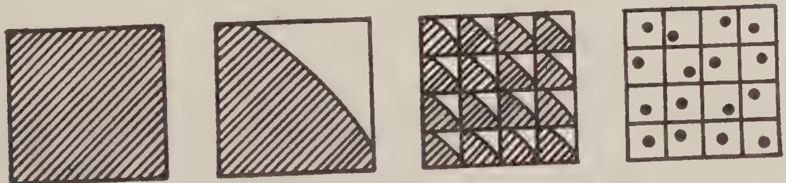
In the preceding sections, the authors have shown the coexistence of the two phases and the horizontal part of the pv-curve from the partition function Ω_N. Surely, Ω_N has a more improved form than the original Ω_i^*. Though the original Ω_i^* (in the hole theory etc.) gives the isotherms of van der Waals type, Ω_N of the whole system reconstructed from Ω_i^* gives the isotherm which has the horizontal part. When the whole system is single phase, however, the reconstruction of Ω_N from Ω_i^* gives no improvement. In this case, $N/m=i_0$, and

$$\frac{1}{N}\log\Omega_N = -\log z + \frac{m}{N}(i_0 \log z + \log \Omega_{i_0}^*)$$
$$= \frac{1}{i_0}\log\Omega_{i_0}^*. \qquad (34)$$

Hence the free energy per molecule suffers no change.

Here we add the relation between the method of the grand partition function and the present one. If we regard each cell as a single system and the whole volume of the vessel as a grandcanonical ensemble, and denote the corresponding grand partition function by $\Xi^*=\Xi^*(T,\tau,\xi)$, the expression $\Xi^*=\sum_{i=0}^{\infty}\Omega_i^* z^i$ holds*. (14) and (15) are nothing but the general formulas of the grand partition function. In this case m_i (Fig. 3) represents the probability of finding a canonical ensemble which consists of cells each of which contains i molecules. We can interpret this probability as the volume fraction of each phase in the vessel. The grand canonical ensemble in Appendix consists of the ensemble of the whole system ($\Xi=\sum_{N=0}^{\infty}\Omega_N z^N$). The results obtained from Ξ are also equivalent to those obtained from Ω_N and Ξ^*.

That is: even if Ω_i^* has an approximate form (giving van der Waals type isotherm), Ω_N, Ξ^* or Ξ reconstructed from it have more improved form as far as we concern to the condensation phenomena.



(a) (b) (c) (d)

Fig. 6. Schematical diagrams for the approximation of the partition function. The hatched region represents the domain of integration in the phase space.

The above mentioned fact, we may say, shows the following fact: the usual partition function heretofore (in the hole theory, etc.) is constructed under the tacit assumption of forced single phase, i.e., the integration in the phase space is performed incompletely. Fig. 6 shows these relations schematically. The square shows the whole phase space, and the hatched domain is the integrated region of the phase integral. (a) The whole integration will give the correct partition function. (b) Approximate partition function Ω_i^* by the hole theory (corresponding to Bethe-Fowler approximation). (c) Ω_N reconstructed from Ω_i^*. (d) Ap-

*) We are much indebted to Mr. S. Ono's valuable remarks.

proximation using "ϵ_i." The discrepancy between the calculation and the experiment in Fig. 5 is due to the roughness of the approximation in (d). However, (c) or even (d) is superior than (b), in the view point that it can give the coexistence of the two phases.

It is the authors' pleasure to wish to express their thanks to Prof. Y. Watanabe and Prof. Y. Takahashi for their kind encouragements and helpful suggestions through the present investigation. We wish also to express our appreciation to the members of Busseiron-Kondankai (the Research Group of Chemical Physics) in Tôhoku University for their discussions.

Appendix

We have replaced the partition function by the greatest term in $\sum^{x}$. We can prove that the results are exactly true in the case $N\to\infty$.

(i) *Method of the grand partition function*

We can easily see that (6) is the coefficient of ζ^N in the expansion of $(\sum_{i=0}^{\infty} x_i\, \tau^i\, \zeta^i)^m$. Hence the grand partition function is

$$\Xi(T, V, \xi)=\sum_{N=0}^{\infty}\Omega(T, V, N, m)\{\lambda\phi(T)\}^N$$

$$=(\sum_{i=0}^{\infty}x_i\xi^i)^m, \tag{35}$$

where $\xi=\lambda\phi(T)\tau=z\tau$.

Then

$$pV/kT=\log\ \Xi=m\log\sum_{i=0}^{\infty}x_i\xi^i, \tag{36}$$

$$\bar{N}=\xi\frac{\partial\ \log\Xi}{\partial\xi}=m\frac{\sum_{i=0}^{\infty}ix_i\xi^i}{\sum_{i=0}^{\infty}x_i\xi^i}\ , \tag{37}$$

where $\bar{N}$ is the mean number of particles under given ξ. (36) and (37) are the same as (14) and (15) except for the difference between $\sum_{i=0}^{N}{}'$ and $\sum_{i=0}^{\infty}$.

(ii) *Method of Kahn and Uhlenbeck.*

From the fact that Ω^N is the coefficient of ζ^N in the expansion of $\left(\sum_{i=0}^{\infty}x_i\tau^i\xi^i\right)^m$ it can be written in the integral representation:

$$\Omega\left(T,\ Nv,\ N,\ \frac{Nv}{\tau}\right)=\frac{1}{2\pi i}\oint\frac{\left(\sum_{i=0}^{\infty}x_i\tau^i\zeta^i\right)^{Nv/\tau}}{\zeta^{N+1}}\,d\zeta. \tag{38}$$

Denoting $S(r)$ by

$$S(r)=\sum^{\infty}\Omega\left(T,\,Nv,\,N,\,\frac{Nv}{\tau}\right)r^N, \tag{39}$$

it can be written as

$$S(r)=\left[1-\frac{v}{\tau}\frac{\sum\limits_{i=0}^{\infty}ix_i\tau^iz^i}{\sum\limits_{i=0}^{\infty}x_i\tau^iz^i}\right]^{-1},$$

where z is defined by

$$z=r\left(\sum_{i=0}^{\infty}x_i\,\tau^iz^i\right)^{v/\tau} \tag{40}$$

for given r.

Let Z be; either (1) the value of z which satisfies

$$\frac{\tau}{v}=\frac{\sum\limits_{i=0}^{\infty}ix_i\tau^iz^i}{\sum\limits_{i=0}^{\infty}x_i\tau^iz^i}, \tag{41}$$

or (2) the smallest singularity of

$$\sum_{i=0}^{\infty}x_i\tau^iz^i, \tag{42}$$

then,

$$\lim_{N\to\infty}\frac{1}{N}\log\Omega\left(T,\,Nv,\,N,\,\frac{Nv}{\tau}\right)=-\log R=-\log\,Z+\frac{v}{\tau}\log\sum_{i=0}^{\infty}x_i\tau^iZ^i,$$

where R is the radius of the convergence of (39). This is equivalent to (13), and the expression of the pressure can be obtained directly.

Here we remark that ξ_{COND} does not appear as the singularity of (42) by the limiting process $N\to\infty$ and $V\to\infty$ under constant v and τ. The process $\tau\to\infty$ is necessary to make ξ_{COND} a singularity.

References

1) J. E. Mayer, J. Chem. Phys. **5** (1937), 67.
2) M. Born and K. Fuchs, Proc. Roy. Soc. London. **A 166** (1938), 391.
3) B. Kahn and G. E. Uhlenbeck, Physica **5** (1938), 399.
4) I. Prigogine's discussion, Convegno Internazionale di Meccanica Statistica, Nuovo Cimento **6** (1949), 225.
5) M. Toda, *Theory of Liquid Structure* (in Japanese) (1947), 148.

Progress of Theoretical Physics, Vol. 5, No. 6, November~December, 1950

On the Mechano-Caloric Effect in Liquid Helium II.

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(Received June 23, 1950)

Recently the second sound velocity of liquid helium II has been measured below 1°K by means of paramagnetic cooling.[1] The result is that the velocity shows a minimum about 1.15°K and then rapidly increases as temperature decreases. This seems at first sight to support Landau's theory[2] rather than Tisza's prediction.[3] The further investigation reveals, however, the fact that Landau's theory will meet with some difficulty, on the other hand, in explaining the mechano-caloric effect. It is the purpose of the present paper to criticize these theories in this connection.

§ 1. Introduction

As is well known, the two-fluid theory of liquid helium II gives the expression for the second sound velocity as follows:[2,3]

$$c_2^2 = s^2 \left(\frac{dT}{ds}\right) \frac{\rho_s}{\rho_n},$$

where s denotes the entropy per unit mass and ρ_n and ρ_s are the normal and superfluid densities respectively. To calculate c_2, Tisza has used the relation

$$\rho_n/\rho = s/s_\lambda, \tag{2}$$

where $\rho=\rho_n+\rho_s$ is the total density and s_λ is the value of s at λ-point. The values of c_2 calculated by using the relation (2) and the experimental data on the entropy agree with the observed values well in the temperature range from λ-point to 1.2°K. But Eq. (1) combined with Eq. (2) predicts generally

$$c_2 \propto T^{1/2} \qquad \text{as} \qquad T \to 0,$$

which seems hardly to be consonant with the recent experimental results.

On the other hand, according to the phonon-roton model of Landau, c_2 increases below 1°K and tends to a finite value at $T=0$. In this case, Tisza's relation (2) is valid only for the roton part at sufficiently low temperatures, while for the phonon part

$$\frac{\rho_n^{(ph)}}{\rho} = \frac{4}{3} \frac{E^{(ph)}}{c_1^2}$$

leaves a constant contribution to c_2, as $E^{(ph)}$ being the energy of the phonon part proportional to T^4 (c_1 the first sound velocity). Clearly the phonon part invalidates Tisza's relation.

From the general point of view, Tisza's relation may be regarded as only a special assumption.[4] We consider, however, this relation to play an essential part in the two-fluid theory as we shall show in the detailed analysis of the mechano-caloric effect, though we recognize also that the phonon part must be taken into account in the low temperature region, where the specific heat has been found experimentally to obey T^3-law.

§ 2. Thermodynamical considerations

F. London has shown that one can derive Tisza's relation phenomenologically without referring to any specific molecular model.[5] His thermodynamical consideration is, however, applicable only to the case of zero Gibbsian potential. We shall generalize his consideration in this respect.

We begin with the experiment of Daunt and Mendelssohn on the mechano-caloric effect. A Dewar flask with a small hole filled by emery powder at the bottom is lifted up from a helium bath above its surface. Then liquid helium II in the flask flows down through the hole at the bottom and, at the same time, the temperature of the liquid inside the flask increases. When the flask is sunk into the bath again and the level of the liquid surface inside the flask becomes equal to that of the bath, its temperature returns also to the original value.

Idealizing this experiment, we consider the following process. Two containers, A and B, filled by liquid helium II are connected by a very narrow capillary, C. The pressure p and the temperature T of the container A are kept constant throughout our process by means of a movable piston and a suitable heat reservoir, while the wall of the container B is assumed to be adiabatic. Thus A, B and C correspond respectively to the helium bath, the Dewar flask and the hole filled by emery powder in the above experiment.

Now, according to the experiment of Kapitza,[6] the flow of liquid helium II through a sufficiently narrow capillary carries no entropy. This fact ensures that A and B can be in equilibrium as far as the following condition is satisfied:[7]

$$g(p', T') = g(p, T), \tag{3}$$

where g is the Gibbsian potential.

Our process starts from the ordinary state of equilibrium where $p'=p$ and $T'=T$. Then we push the piston of B and increase its pressure p' by a small amount. An amount of liquid helium II flows from B to A through C. At the same time the temperature of B rises and a new state of equilibrium will be established when the condition (3) is satisfied. Now, we increase the pressure of B continuously, but so slowly that the temperature rising follows the pressure increasing and liquid helium will flow on from B to A with infinitesimal rate. In this process, Eq. (3) may be satisfied at each step and, the right hand side of it remaining constant, the pressure and temperature of B increase according to the equation of H. London:

$$dp'/dT'=\rho's'.$$

However, the representative point, (p', T'), in the pT-diagram reaches sooner or later a point, (p_λ, T_λ), on the λ-line, where liquid helium loses its superfluidity and can no more flow through C.

Let M and M_λ be the masses of liquid helium contained in B at the beginning and the end of our process respectively. Then the experimental fact proved by Kapitza is expressed as follows:

$$Ms(p, T)=M_\lambda s(p_\lambda, T_\lambda).$$

This is just the relation of Tisza, (2), for the mass of the normal fluid may be considered to be given by the mass M_λ which did not take part in the superfluid flow. As the λ-line is practically parallel to the p-axis and the entropy is experimentally found to be almost independent of pressure, $s_\lambda=s(p_\lambda, T_\lambda)$ may be regarded as a constant.

Thus we have accomplished a phenomenological definition of the normal fluid density on the basis of the experimental fact that there exists a superfluid part in the capillary flow of liquid helium II which, moreover, carries no entropy. Starting from this definition, we can construct the two-fluid theory of liquid helium II, as one of the present authors has shown elsewhere.[8] In the above derivation of Tisza's relation, we have tacitly assumed that the normal fluid mass should be conserved during adiabatic reversible processes. The two-fluid theory is formulated self-consistently with respect to this assumption, for we can derive the following equation from this theory:

$$\int \Gamma d\tau=\frac{1}{s_\lambda}\left[\int\left(\frac{ds}{dt}\right)_{irr} d\tau + \int \frac{q\cdot d\sigma}{T}\right],$$

where Γ is the rate of transformation from the superfluid to the normal fluid per unit volume, $(ds/dt)_{irr}$ is the rate of irreversible entropy production per unit volume and $q\cdot d\sigma$ is the heat current through surface element $d\sigma$.

Presumably we may conclude that the normal fluid mass is not conserved during adiabatic reversible processes in Landau's theory. But this conclusion contradicts the fundamental equations of the second sound in his theory.

§ 3. The condensed Bose-Einstein gas

In the preceding section, we have assumed that the normal fluid mass should be conserved in adiabatic compression or expansion. This seems at first sight somewhat peculiar, for, in an ordinary gas, adiabatic compression results in temperature increasing so that a new distribution of molecules appropriate to the increased temperature will be established. Another example of such exceptionary property is, however, found in the case of the condensed Bose-Einstein perfect gas.[9] In this case, the number of molecules in excited energy levels is given by

$$N_n=2.612\left[\frac{2\pi\, m^2 k}{h^2}\right]^{3/2}\cdot\frac{N}{\rho}T^{3/2},$$

and tne remaining N-N_n molecules are all condensed at the lowest level in momentum space. Here, k is Boltzmann's constant. Only the molecules in excited levels contribute to the entropy which is given by

$$S=1.341\left[\frac{2\pi\, m^2 k}{h^2}\right]^{3/2}\cdot\frac{5N}{2\rho}kT^{3/2}.$$

Accordingly, using the entropy at the condensation point

$$S_0=\frac{5}{2}\cdot\frac{2.612}{1.341}Nk,$$

we can derive "Tisza's relation":

$$N_n/N=S/S_0. \tag{4}$$

Now, we imagine that we push the molecules into a box filled by the condensed B.E. gas one by one through a very small hole. It is assumed that the density of the gas is kept constant by expanding the volume and no heat is supplied during the process. From the adiabatic condition, we get

$$dS=\text{const. } T^{3/2}\left\{dN+\frac{3}{2}NT^{-1}dT\right\}=0,$$

which gives the relation between the increase of the number of molecules and the decrease of the temperature. Inserting this relation into the expression for the variation of the number of excited molecules,

$$dN_n=\text{const.} T^{3/2}\left\{dN+\frac{3}{2}NT^{-1}dT\right\},$$

we conclude that the molecules pushed into the box should condense to the lowest level so that the number of excited molecules remains unaltered during the adiabatic expansion. Eq. (4) expresses this fact in integrated form.

Practically it is impossible to push the molecules into the box one by one. In the case of liquid helium II, the superfluid flow through a narrow capillary has played this part.

We are indebted to Mr. T. Usui for the valuable conversation with him.

References

1) J. R. Pellam and R. B. Scott, Phys. Rev. **76** (1949), 869; R. D. Maurer and M. A. Herlin, ibid. **76** (1949), 948.
2) L. Landau, J. Phys. USSR. **5** (1941), 71.
3) L. Tisza, Phys. Rev. **72** (1947), 838.
4) C. J. Gorter, Physica **15** (1949), 523.
5) F. London, Rev. Mod. Phys. **17** (1945), 310
6) P. Kapitza, Phys. Rev. **60** (1941), 354.
7) A. Harashima, Busseiron-Kenkyu (in Japanese) **18** (1949), 71.
8) S. Nakajima, K. Tomita and T. Usui, Phys. Rev. **78** (1950) 768.
9) T. Nagamiya, *Liquid Helium* (in Japanese) (1947).

Progress of Theoretical Physics, Vol. 5, No. 6, November~December, 1950

Radiative Correction to Decay Processes, II

—Beta Disintegration of Nucleon—*

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(Received June 29, 1950)

§ 1. Introduction

In recent years a method has been developed by Tomonaga,[1] Schwinger,[2] Feynman,[3] and Dyson[4] to remove self-consistently the divergence difficulties in the quantum electrodynamics with the introduction of the idea of mass and charge renormalizations. Stimulated by this success, applications of this self-consistent subtraction method have been tried by many physicists to other more general systems than that of electron and electromagnetic field, yielding finite results for many processes which had hitherto suffered from divergences. However, there are reported negative examples also, where divergences involved in the reaction of one field on the other can not be concealed in mass and charge (more generally, interaction constant) :—as, the anomalous magnetic moment of nucleon due to vector meson with tensor coupling,[5] the induced electric current for vector meson due to fluctuation of radiation field and vacuum polarization,[6] etc. Therefore, the applicability of this method seems to be rather restricted.

Here arises a question whether the self-consistent subtraction method would be successful when applied to systems consisting of more than two mutually interacting wave fields, if it is the case for each sub-system composed of two of the constituents. From such a point of view we have treated the problem of the radiative correction to decay processes. In the first part of this paper[7] two of us reported the result for the decay of a scalar meson interacting with light particles through scalar coupling. The answer was in that case affirmative apart from the lack in uniqueness of the renormalization of the coupling constant.

In the present paper we deal with the second order radiative correction to the beta disintegration of nucleon,[8] assuming the Fermi's interaction between necleon and lepton fields, and show that there appears such an ultraviolet divergence as can not be removed by the procedure of renormalization of mass and coupling constant, so far as a single type of coupling is assumed for the decay Hamiltonian. The possibility of compensation of divergences by mixing several types of coupling is also discussed.

§ 2. Effective Hamiltonian for beta disintegration

We start from the equation in the Tomonaga-Schwinger's covariant formalism of the form

$$\{H(x)+V(x)-i\partial/\partial\sigma(x)\}\chi[\sigma]=0, \tag{2.1}$$

where $H(x)$ is the Hamiltonian describing the interaction of radiation field with proton and electron fields, and $V(x)$ that accounting for beta disintegration.

$H(x)$ is, provided that the mass renormalization has been done from the beginning for electron as well as for proton, given by

$$\begin{aligned} H(x)=&ieA_\mu(x)\bar{\psi}(x)\gamma_\mu\psi(x)-\delta m\bar{\psi}(x)\psi(x) \\ &-ieA_\mu(x)\bar{\Psi}(x)\Gamma_\mu\Psi(x)-\delta M\bar{\Psi}(x)\Psi(x), \end{aligned} \tag{2.2}$$

where $A_\mu(x)$ describes radiation field, $\psi(x)$ and $\bar{\psi}(x)$ electron field, $\Psi(x)$ and $\bar{\Psi}(x)$ proton field, and δm and δM denote the electromagnetic masses of electron and proton, respectively. γ_μ's are the Dirac matrices for lepton, while Γ_μ's are used to designate those for nucleon.

$V(x)$ has, according to Bethe, five possible types which may generally be written as

$$V(x)=g\bar{\Phi}(x)\bar{\varphi}(x)\beta B\psi(x)\Psi(x)+\text{conj.}, \tag{2.3}$$

where $\Phi(x)$ and $\bar{\Phi}(x)$ describe the neutron wave, $\varphi(x)$ and $\bar{\varphi}(x)$ the neutrino wave, g denoting the coupling constant. β and B are the sedinions characterizing the type of coupling, given in the following Table.

Table

Type of coupling	Scalar	Vector	Tensor	Pseudo-vector	Pseudo-scalar
g	g_s	g_v	g_t	g_{pv}	g_{ps}
β	1	γ_μ	$\sigma_{\mu\nu}=\frac{1}{2i}(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)$	$i\gamma_5\gamma_\mu$	γ_5
B	1	Γ_μ	$\Sigma_{\mu\nu}$	$i\Gamma_5\Gamma_\mu$	Γ_5
βB	1	$\gamma_\mu\Gamma_\mu$	$\sigma_{\mu\nu}\Sigma_{\mu\nu}$	$-\gamma_5\Gamma_5\gamma_\mu\Gamma_\mu$	$\gamma_5\Gamma_5$
$\beta B\sigma_{\mu\nu}\Sigma_{\mu\nu}$	$\sigma_{\mu\nu}\Sigma_{\mu\nu}$	$-6(1+\gamma_5\Gamma_5)\cdot\gamma_\mu\Gamma_\mu$	$24(1+\gamma_5\Gamma_5)-8\sigma_{\mu\nu}\Sigma_{\mu\nu}$	$6(1+\gamma_5\Gamma_5)\cdot\gamma_\mu\Gamma_\mu$	$\sigma_{\mu\nu}\Sigma_{\mu\nu}$

The wave functions, of course, satisfy the usual wave equations and commutation relations for free fields, of which the mass parameters are to be taken for observed ones, since they are assumed to have already been renormalized.

The effective Hamiltonian for beta decay under the influence of the reaction of radiation field is now given by $V_T(x)=U^{-1}[\sigma]V(x)U[\sigma]$, with $U[\sigma]$ as a

unitary transformation functional that satisfies the equation $\{H(x)-i\partial/\partial\sigma(x)\}U[\sigma]=0$ and is subjected to the initial condition $U[-\infty]=1$. Using this effective Hamiltonian, we can evaluate the total rate of beta decay, to the second order of approximation in g but inclusively of the radiative correction to any desired order in e, in accordance with the Schwinger's formula for transition probability, which reads

$$w=\int d^3x\int d^4x'(1\mid V_T(x)V_T(x')\mid 1), \tag{2.4}$$

where the matrix labels, 1's, designate the initial state of the system considered.

Let us confine ourselves to e^2 approximation. Then w may be split into three parts as

$$w=w_0+\delta_0 w+\delta_1 w, \tag{2.5}$$

with

$$w_0=\int d^3x\int d^4x'(1\mid V(x)V(x')\mid 1), \tag{2.6}$$

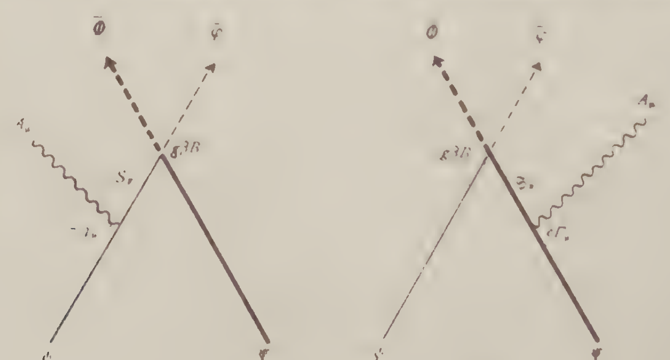
$$\delta_0 w=\int d^3x\int d^4x'(1\mid V(x)V_T^{(2)}(x')+V_T^{(2)}(x)V(x')\mid 1), \tag{2.7}$$

and

$$\delta_1 w=\int d^3x\int d^4x'(1\mid V_T^{(1)}(x)V_T^{(1)}(x')\mid 1) \tag{2.8}$$

where $V_T^{(1)}(x)$ and $V_T^{(2)}(x)$ are respectively the first and the second order terms of the power series in e of $V_T(x)$. w_0 gives the inverse of the life time of beta decay without radiative correction, and $\delta_0 w$ describes the correction on the rate of radiationless decay due to emission and reabsorption of virtual photons, while $\delta_1 w$ accounts for decay accompanied by emission of a real photon.

For actual calculation it is more convenient to use, instead of $V_T(x)$, the Feynman's effective Hamiltonian, $V_F(x)$, which is, following Dyson, practically



(1a) (1b)

Fig. 1

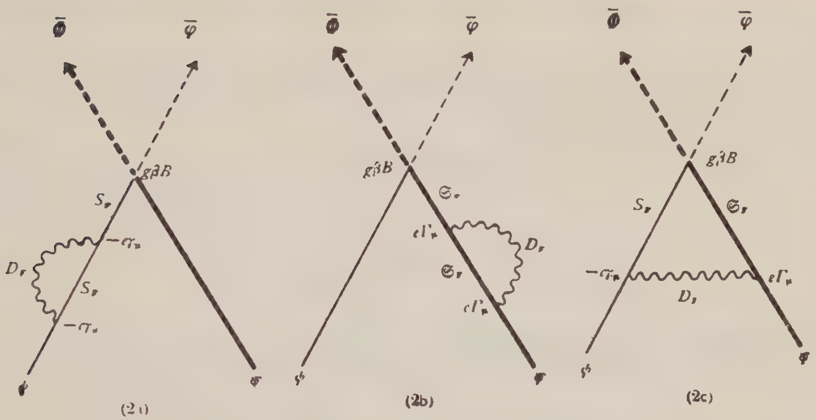


Fig. 2

equivalent and related to $V_r(x)$ through $V_r(x)=U[\infty]V_r(x)$. The expression for the first and the second order terms in e of $V_r(x)$ can be obtained at once with aid of the Feynman-Dyson's diagrams given in Fig. 1 and Fig. 2, respectively. The results are

$$\begin{aligned}V_r^{(1)}(x)=&ieg\bar{\Phi}(x)\bar{\varphi}(x)\beta B\int d^4x' S_r(x-x')\gamma_\mu\psi(x')A_\mu(x')\Psi(x)\\&-ieg\bar{\Phi}(x)\bar{\varphi}(x)\beta B\psi(x)\int d^4x'\mathfrak{S}_r(x-x')\Gamma_\mu\Psi(x')A_\mu(x')\\&+\text{conj.},\end{aligned}\tag{2.9}$$

$$\begin{aligned}V_r^{(2)}(x)=&g\bar{\Phi}(x)\bar{\varphi}(x)\beta B\{\psi_r^{(2)}(x)\Psi(x)+\psi(x)\Psi_r^{(2)}(x)+W(x)\}\\&+\text{conj.},\end{aligned}\tag{2.10}$$

where

$$\begin{aligned}\psi_r^{(2)}(x)=&\int d^4x' S_r(x-x')\{-\delta m\psi(x')\\&+ie^2\int d^4x''\gamma_\mu S_r(x'-x'')\gamma_\mu\psi(x'')D_r(x'-x'')\},\end{aligned}\tag{2.11}$$

$$\begin{aligned}\Psi_r^{(2)}(x)=&\int d^4x'\mathfrak{S}_r(x-x')\{-\delta M\Psi(x')\\&+ie^2\int d^4x''\Gamma_\mu\mathfrak{S}_r(x'-x'')\Gamma_\mu\Psi(x'')D_r(x'-x'')\},\end{aligned}\tag{2.12}$$

and

$$W(x)=-ie^2\int d^4x'\int d^4x'' S_r(x-x')\gamma_\mu\psi(x')D_r(x'-x'')\mathfrak{S}_r(x'-x'')\Gamma_\mu\Psi(x''),\tag{2.13}$$

discarding terms which contribute nothing to the decay probabilities. In the above expressions, $S_r(x)$ and $D_r(x)$ are the same functions as those introduced in the

Dyson's papers except that the sign of the argument is reversed. $\mathfrak{S}_P(x)$ refers to proton.

The integral (2.11) has already been evaluated by Schwinger and by Karplus and Kroll,[9] and reduces to the form

$$\psi_F^{(2)}(x)=z^{(2)}\psi(x), \tag{2.14}$$

with

$$z^{(2)}=\frac{ie^2}{(2\pi)^4}\lim_{\epsilon\to+0}\int d^4k\int_0^1 udu\frac{k^2-m^2(4-4u-2u^2)}{(k^2+m^2u^2-i\epsilon)^3}, \tag{2.15}$$

in which k represents the 4-momentum of a virtual photon, m being the electron mass. With the k and u integration performed, (2.15) becomes

$$z^{(2)}=\frac{\alpha}{2\pi}\left(\log\frac{m}{2k_{\min}}-\frac{1}{2}\log\frac{2k_{\max}}{m}+\frac{1}{8}\right), \tag{2.16}$$

where a lower and a higher cut-off frequencies, $k_{\min}$ and $k_{\max}$, have been introduced to express divergent integrals. α is the fine structure constant, $e^2/4\pi$.

In quite the same way,

$$\Psi_F^{(2)}(x)=Z^{(2)}\Psi(x). \tag{2.17}$$

The expression for $Z^{(2)}$ can be obtained at once by replacing m in (2.16) with the proton mass, M.

We must now evaluate $W(x)$, which is expressed in terms of Fourier integral as

$$W(x)=\int d^4p\int d^4P\,K(p,\gamma\,;P,\Gamma)e^{ipx}\psi(p)e^{iPx}\Psi(P), \tag{2.18}$$

with

$$K(p,\gamma\,;P,\Gamma)=\lim_{\epsilon\to+0}\frac{-ie^2}{(2\pi)^4}\int d^4k\frac{(i\gamma(p+k)-m)\gamma_\mu(i\Gamma(P-k)-M)\Gamma_\mu}{((p+k)^2+m^2-i\epsilon)((P-k)^2+M^2-i\epsilon)(k^2-i\epsilon)}. \tag{2.19}$$

Following Feynman and Schwinger, we transform this into the form

$$K(p,\gamma\,;P,\Gamma)=\lim_{\epsilon\to+0}\frac{-ie^2}{(2\pi)^4}\int d^4k\int_0^1 udu\int_{-0}^1 dv\frac{1}{(k^2+u^2\lambda^2-i\epsilon)^3}\cdot$$

$$\cdot\{(i\gamma(p-uR)-m)\gamma_\mu(i\Gamma(P+uR)-M)\Gamma_\mu+\frac{k^2}{4}\gamma_\mu\gamma_\nu\Gamma_\mu\Gamma_\nu\}, \tag{2.20}$$

putting

$$R\equiv R(v\,;p,P)=\frac{1+v}{2}p-\frac{1-v}{2}P, \tag{2.21}$$

$$\lambda^2\equiv\lambda^2(v,pP)=-R^2=\left(\frac{1+v}{2}\right)^2m^2+\left(\frac{1-v}{2}\right)^2M^2+\frac{1-v^2}{2}pP. \tag{2.22}$$

Then, using the formulae

$$\lim_{\epsilon\to+0}\int d^4k\frac{1}{(k^2+m^2-i\epsilon)^2}=2\pi^2 i\left(\log\frac{2k_{\max}}{m}-1\right),$$

$$\lim_{\epsilon\to+0}\int_0^1 udu\int d^4k\frac{u^n}{(k^2+u^2m^2-i\epsilon)^3}=\begin{cases}\dfrac{i\pi^2}{2m^2}\left(\log\dfrac{m}{2k_{\min}}+1\right) & \text{for } n=0,\\[2ex] \dfrac{i\pi^2}{2nm^2} & \text{for } n=1,\ 2,\ 3,\ \ldots,\end{cases} \tag{2.23}$$

we get

$$\begin{aligned}K(p,\gamma\,;P,\Gamma)=&\frac{\alpha}{4\pi}\left\{\log\frac{2k_{\max}}{m}+\log\frac{2k_{\max}}{M}+\frac{5}{2}+\frac{M^2-m^2}{2}F_1-\frac{M^2+m^2+2pP}{2}F_0\right\}\\&\times\frac{\gamma_\mu\gamma_\nu\Gamma_\mu\Gamma_\nu}{4}-\frac{\alpha}{4\pi}\left\{\left(\log\frac{m}{2k_{\min}}+\log\frac{M}{2k_{\min}}\right)F_0+G\right\}pP\\&+\frac{\alpha}{64\pi}\Big\{8(F_0+F_1)[m^2+(\gamma p)(\gamma P)]+8(F_0-F_1)[M^2+(\Gamma P)(\Gamma p)]\\&\qquad-(F_0-F_2)[(\gamma p)(\Gamma P)+(\gamma P)(\Gamma p)](\gamma\Gamma)\\&+(F_0+2F_1+F_2)(\gamma p)(\Gamma p)(\gamma\Gamma)+(F_0-2F_1+F_2)(\gamma P)(\Gamma P)(\gamma\Gamma)\Big\},\end{aligned} \tag{2.24}$$

where F_n and G are defined by

$$F_n\equiv F_n(pP)=\int_{-1}^1 dv\frac{v^n}{\lambda^2(v,\,pP)}\qquad(n=0,1,2,\ldots),$$

and

$$G\equiv G(pP)=\int_{-1}^1 dv\frac{1}{\lambda^2(v,\,pP)}\log\frac{\lambda^2(v,\,pP)}{mM}. \tag{2.25}$$

Thus, combining the above results for $z^{(2)}$, $Z^{(2)}$ and $K(p,\gamma\,;P,\Gamma)$, we obtain for the second order effective Hamiltonian the following expression

$$V_F^{(2)}(x)=V_F^{(2)}(x)_{\text{ultra}}+V_F^{(2)}(x)_{\text{infra}}+V_F^{(2)}(x)_{\text{finite}}, \tag{2.26}$$

with

$$\begin{aligned}V_F^{(2)}(x)_{\text{ultra}}=&-\frac{\alpha}{16\pi}\left(\frac{2k_{\max}}{m}+\log\frac{2k_{\max}}{M}\right)\\&\times g\bar{\Phi}(x)\bar{\varphi}(x)\beta B\sigma_{\mu\nu}\Sigma_{\mu\nu}\psi(x)\Psi(x)+\text{conj.},\end{aligned} \tag{2.27}$$

and

$$\begin{aligned}V_F^{(2)}(x)_{\text{infra}}=&\frac{\alpha}{4\pi}\left(\log\frac{m}{2k_{\min}}+\log\frac{M}{2k_{\min}}\right)\\&\times g\bar{\Phi}(x)\bar{\varphi}(x)\beta B(2-pPF_0)\psi(x)\Psi(x)+\text{conj}.\end{aligned} \tag{2.28}$$

Here we have omitted writing down a lengthy expression for the finite term, $V_I^{(2)}(x)_{\text{finite}}$. The relation

$$4-\gamma_\mu\gamma_\nu\Gamma_\mu\Gamma_\nu=\sigma_{\mu\nu}\Sigma_{\mu\nu} \tag{2.29}$$

has been used in deriving (2.27). p_μ and P_μ in (2.28) are to be interpreted as the differential operator, $-i\partial/\partial x_\mu$, operating on $\psi(x)$ and on $\Psi(x)$, respectively.

The two divergent terms, (2.27) and (2.28), will be discussed in the succeeding sections.

§ 3. Infrared divergence

From the general considerations on the infrared catastrophe it is expected that the infrared divergence involved in the virtual photon processes will present no difficulty but serve to cancel that which appears in the real photon processes.[10] In this section we shall confirm that this is actually the case, confining ourselves to the case of neutron decay for simplicity.

Let us first evaluate the contribution of $V_I^{(2)}(x)_{\text{infra}}$ to the rate of beta decay. This is obtained by substituting (2.28) into (2.7), using the following Fourier integral representations for the vacuum expectation values, and the one neutron expectation value averaged over two possible spin orientations, of bilinear products of wave functions:

$$\langle\psi(x)\bar{\psi}(x')\rangle_0=-(2\pi)^{-3}\int_{p_0>0}d^4p\delta(p^2+m^2)(i\gamma p-m)e^{ip(x-x')},$$

$$\langle\bar{\varphi}(x)O\varphi(x')\rangle_0=-(2\pi)^{-3}\int_{q_0<0}d^4q\delta(q^2+n^2)S_p[O(i\gamma q-n)]e^{iq(x'-x)},$$

$$\langle\Psi(x)\bar{\Psi}(x')\rangle_0=-(2\pi)^{-3}\int_{P_0>0}d^4P\delta(P^2+M^2)(i\Gamma P-M)e^{iP(x-x')}, \tag{3.1}$$

and

$$\frac{1}{2}\sum_{\text{spin}}\langle\Phi(x)O\bar{\Phi}(x')\rangle_1=-\frac{1}{4Q_0V_3}S_p[O(i\Gamma Q-N)]e^{iQ(x'-x)},$$

where V_3 stands for the volume of the whole three dimensional space in which the neutron is assumed to be at rest before decay, n and N denoting the neutrino mass and the neutron mass, respectively. The result can be written as

$$\delta_0 w_{\text{infra}}=-\frac{ag^2}{4Q_0(2\pi)^6}\left(\log\frac{m}{2k_{\min}}+\log\frac{M}{2k_{\min}}\right)\int_{p_0>0}d^4p\delta(p^2+m^2)$$

$$\times\int_{P_0>0}d^4P\delta(P^2+M^2)\int_{q_0<0}d^4q\delta(q^2+n^2)\delta(Q+q-P-p)S_p[\beta(i\gamma p-m)$$

$$\times\beta'(i\gamma q-n)S_p[B(i\Gamma P-M)B'(i\Gamma Q-N)](2-pPF_0), \tag{3.2}$$

where $Q=(0, 0, 0, iN)$ is the 4-momentum of the neutron, and p, $-q$, and P

represent those of the decay products, i.e.. of an electron, and an antineutrino, and a proton, respectively.

The rate of neutron decay that is accompanied by emission of a real photon can be evaluated from (2.8) and (2.9) in a similar manner, using

$$\langle A_\mu(x)A_\nu(x')\rangle_0=\delta_{\mu\nu}(2\pi)^{-3}\int_{k_0>0} d^4k\delta(k^2)e^{ik(x-x')} \tag{3.3}$$

besides (3.1). The result is

$$\begin{aligned}\delta_1 w=&\frac{\alpha g^2}{8Q_0(2\pi)^7}\int_{p_0>0} d^3p\delta(p^2+m^2)\int_{P_0>0} d^4P\delta(P^2+M^2)\int_{q_0>0} d^4q\delta(q^2+n^2)\\ &\times\int_{k_0>0} d^4k\delta(k^2)\delta(Q+q-P-p-k)\{(Pk)^{-2}S_p[\beta(i\gamma p-m)\beta'(i\gamma q-n)]\\ &\times S_p[B(i\Gamma(P+k)-M)\Gamma_\mu(i\Gamma P-M)\Gamma_\mu(i\Gamma(P+k)-M)B'(i\Gamma Q-N)]\\ &+(pk)^{-2}S_p[\beta(i\gamma(p+k)-m)\gamma_\mu(i\gamma-m)\gamma_\mu(i\gamma(p+k)-m)\beta'(i\gamma q-n)]\\ &\times S_p[B(i\Gamma P-M)B'(i\Gamma Q-N)]\\ &-2(pk)^{-1}(Pk)^{-1}S_p[\beta(i\gamma p-m)\gamma_\mu(i\gamma(p+k)-m)\beta'(i\gamma q-n)]\\ &\times S_p[B(i\Gamma(P+k)-M)\Gamma_\mu(i\Gamma P-M)B'(i\Gamma Q-N)]\},\end{aligned} \tag{3.4}$$

in which k is this time the 4-momentum of a real photon.

The above k integral converges in the high frequency region, but diverges near the low frequency end. Its behavior of divergence can most easily be seen by putting $k=0$ in $\delta(Q+q-P-p-k)$ and the traces:

$$\begin{aligned}\delta_1 w_{\text{infra}}=&\frac{\alpha g^2}{4Q_0(2\pi)^6}\int_{p_0>0} d^4p\delta(p^2+m^2)\int_{P_0>0} d^4P\delta(P^2+M^2)\int_{q_0>0} d^4q\delta(q^2+n^2)\\ &\times\delta(Q+q-P-p)S_p[\beta(i\gamma p-m)\beta'(iq-n)]S_p[B(i\Gamma P-M)B'(i\Gamma Q-N)]\\ &\times\frac{1}{\pi}\int_{k_0>0} d^4k\delta(k^2)\left\{\frac{M^2}{(Pk)^2}+\frac{m^2}{(pk)^2}+\frac{2pP}{(pk)(Pk)}\right\},\end{aligned} \tag{3.5}$$

of which the last k integral can be written, after the integration over angles, in the form

$$\frac{1}{\pi}\int_{k_0>0} d^4k\delta(k^2)\left\{\frac{M^2}{(Pk)^2}+\frac{m^2}{(pk)^2}+\frac{2pP}{(pk)(Pk)}\right\}=2\int_0^{+\infty}\frac{dk}{k}(2-pPF_0). \tag{3.6}$$

On comparing (3.5) and (3.6) with (3.2), it turns out that the infrared divergence disappears from the total radiative correction.

§ 4. Ultraviolet divergence

From (2.3) ond (2.27) it is noticed that $V_F^{(2)}(x)_{\text{ultra}}$ contains an extra sedinion, $\sigma_{\mu\nu}\Sigma_{\mu\nu}$, as compared with $V(x)$. This sedinion brings about change in the

type of coupling, as shown in the lowest row of the Table in § 2. Consequently, the ultraviolet divergence involved in $V_F^{(2)}(x)$ can not be removed with the use of the idea of renormalization of the coupling constant, g, so far as a single type is assumed for the coupling between lepton and nucleon.

There is, however, a possibility of removing the divergence with, or even without, the use of the idea of g-renormalization, if it is allowed to mix several types of coupling.

Consider that the Hamiltonian for decay is given by

$$V(x)=\bar{\Phi}(x)\bar{\varphi}(x)\sum g_i\beta B\psi(x)\Psi(x)+\text{conj.}, \tag{4.1}$$

with

$$\sum g_i\beta B=g_s+g_v\gamma_\mu\Gamma_\mu+g_t\sigma_{\mu\nu}\Sigma_{\mu\nu}-g_{pv}\gamma_5\Gamma_5\gamma_\mu\Gamma_\mu+g_{ps}\gamma_5\Gamma_5 \tag{4.2}$$

instead of (2.3). Then we shall have

$$V_F^{(2)}(x)_{\text{ultra}}=\bar{\Phi}(x)\bar{\varphi}(x)C\sum g_i\beta B\sigma_{\mu\nu}\Sigma_{\mu\nu}\psi(x)\Psi(x)+\text{conj.}, \tag{4.3}$$

with

$$\begin{aligned}\sum g_i\beta B\sigma_{\mu\nu}\Sigma_{\mu\nu}=24g_t(1+\gamma_5\Gamma_5)+6(g_{pv}-g_v)(1+\gamma_5\Gamma_5)\gamma_\mu\Gamma_\mu\\ +(g_s+g_{ps}-8g_t)\sigma_{\mu\nu}\Sigma_{\mu\nu}\end{aligned} \tag{4.4}$$

in place of (2.27), putting for brevity

$$C=-\frac{a}{16\pi}\left(\log\frac{2k_{\max}}{m}+\log\frac{2k_{\max}}{M}\right). \tag{4.5}$$

If we take a viewpoint that $V_F^{(2)}(x)_{\text{ultra}}$ can be ascribed to the modifications of the coupling constants, it follows from comparison of (4.1) and (4.2) with (4.3) and (4.4) that they should be chosen as below:

$$\begin{aligned}\delta g_s&=\delta g_{ps}=24Cg_t,\\ \delta g_t&=C(g_s+g_{ps}-8g_t),\end{aligned} \tag{4.6}$$

and

$$\delta g_v=-\delta g_{pv}=6C(g_v-g_{pv}). \tag{4.7}$$

These relations indicate that the five types of coupling can be classified into two independent sets, members of each set being mutually interrelated: (i) The vector and the pseudovector couplings constitute one set, while (ii) the scalar, the pseudoscalar, and the tensor couplings belong to the other. Let us, therefore, assign a single coupling constant to each set and seek the weights that must be attached to its members so as to remove the divergence difficulty with, or without, the procedure of renormalization of the coupling constants.

The results are as follows:

(ia) $g_v=g_{pv}\equiv f,$ $\delta f=0,$

$$
\begin{array}{llll}
\text{(ib)} & g_v = -g_{pv} \equiv f, & \delta f = 12Cf, & \\
\text{(iia)} & g_s = -g_{ps} \equiv g,\ g_t = 0, & \delta g = 0, & (4.8) \\
\text{(iib)} & g_s = g_{ps} = 6g_t \equiv g, & \delta g = 4Cg, & \\
\text{(iic)} & g_s = g_{ps} = -2g_t \equiv g, & \delta g = -12Cg, &
\end{array}
$$

of which the first two refer to the set of the vector and the pseudovector couplings, while the remaining three to the set of scalar, the pseudoscalar, and the tensor couplings.

It is, however, to be noted that the renormalization of the coupling constant lacks in uniqueness, like in the case of meson decay dealt with in the first part of this paper.

The only mixtures that require no renormalization of the coupling constant are given by the following three combinations of (ia) with (iia) :

$$
\begin{array}{lllll}
\text{(i)} & g_v = g_{pv} \equiv f, & g_s = g_{ps} = g_t = 0, & & \\
\text{(ii)} & g_v = g_{pv} = 0, & g_s = -g_{ps} \equiv g, & g_t = 0 & (4.9) \\
\text{(iii)} & g_v = g_{pv} \equiv f, & g_s = -g_{ps} \equiv g, & g_t = 0. &
\end{array}
$$

But there is no experimental evidence that supports these mixtures of couplings.

References

*) A preliminary report of the main part of this work has been published in the letter to the editor column of this journal, **5** (1950), 338. In that letter we have made a slight mistake in the last several lines, which describe the g-renormalization in the case of mixture of the scalar, the pseudoscalar, and the tensor couplings. The mistake is corrected in the present paper.

1) T. Tati and S. Tomonaga, Prog. Theor. Phys. **3** (1948), 391; H. Fukuda, Y. Miyamoto, and S. Tomonaga, Prog. Theor. Phys. **4** (1949), 47, 121. For papers published by Tomonaga and his coworkers before 1948 refer to S. Tomonaga, Phys. Rev. **74** (1948), 224.
2) J. Schwinger, Phys. Rev. **74** (1948), 1439; **75** (1949), 651; **76** (1949), 790.
3) R. P. Feynman, Phys. Rev. **76** (1949), 749, 769.
4) F. J. Dyson, Phys. Rev. **75** (1949), 486, 1736.
5) K. M. Case, Phys. Rev. **76** (1949), 1.
6) T. Kinoshita and Y. Nambu, Prog. Theor. Phys. **5** (1950), 307.
7) S. Hanawa and T. Miyazima, Prog. Theor. Phys. **5** (1950), 459.
8) As for electromagnetic radiation accompanying beta disintegration of nuclei see F. Bloch, Phys. Rev. **50** (1936), 272; J. K. Knipp and G. Uhlenbeck, Physica **3** (1936), 425.
9) R. Karplus and N. M. Kroll, Phys. Rev. **77** (1950), 536.
10) F. Bloch and A. Nordsieck, Phys. Rev. **52** (1937), 54; Z. Koba Prog. Theor. Phys. **3** (1948), 208.

Progress of Theoretical Physics, Vol. 5, No. 6, November~December, 1950

Ambiguities in Quantized Field Theories

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(Received July 17, 1950)

§ 1. Introduction

Recent development of the Tomonaga-Schwinger theory enabled us to calculate various processes in a completely covariant manner. As is well-known, however, it inherits the divergence difficulties of older theories without any improvement, which come from the singular behavior of D-functions employed. Furthermore, we are often encountered with cases in which results of computation contradict with such formal requirements as gauge invariance, divergence theorem, and equivalence theorem[1]. Though closely related to the divergence difficulty, the latter has some special feature which makes it distinguished from the former, and therefore is treated separately under the name of *ambiguity* of the present field theories with which we shall be concerned in this note.

This difficulty has bothered many physicists from the early stage of the Tomonaga-Schwinger theory and various efforts have been made to remove or circumvent it. One of them is the regularization method proposed by Pauli and Villars[2] which is effective in eliminating any undesirable term under suitable conditions. It seems to be unreliable, however, since it has no definite rule in itself to set up such conditions[3], even if we disregard the fact that it does contradict with concepts of present field theories. Another trial is one which consists in mixing several fields under some conditions.[4] These fields are supposed to interact themselves with real coupling constants. Although favored by the problem of photon self-energy, this method will not be powerful in removing ambiguities in other general cases. To make the matter worse, it is most probable that the mixing procedure does not stop and close by finite steps and further it cannot be valid any more in higher order approximations than the second.

Recently, Katayama[5] treated this problem from a different point of view. He wants to remove the ambiguity not through drastic modification of the usual theory but merely assigning appropriate values to some integrals of singular functions. This method is surely elementary, but it seems to be more instructive and profitable than others to take the complication of our present knowledge into order. Though he was successful within the range of his investigation, his result seems to be devoid of enough rational foundation, since his conditions were not derived from general consideration and he only tested their usefulness with some

(not all) examples. It is the aim of our investigation to grasp the problem in its general feature and find out whether his conditions are sufficient or not.

The problem of ambiguity occurs most frequently when several bosons interact through virtual fermion fields*. The representative process is that described by a Feynman-Dyson diagram which consists of a closed fermion loop with $(n+1)$-vertices on it and $(n+1)$-boson lines running out from them. The corresponding matrix is

$$M=\int_{-\infty}^{\infty} dx_0 dx_1 \cdots dx_n U_0(x_0) U_1(x_1) \cdots U_n(x_n)$$
$$\times S_p\{ \sum_{j=0}^{n} \bar{S}(x_0-x_1)\Gamma_1 \bar{S}(x_1-x_2)\Gamma_2 \cdots \Gamma_j\ S^{(1)}(x_j-x_{j+1})\Gamma_{j+1} \cdots \Gamma_n \bar{S}(x_n-x_0)\Gamma_0\} \tag{1}$$

where $U_0(x_0), \cdots, U_n(x_n)$ represent potentials of boson fields or their space-time derivatives and Γ_m's mean 1, γ_μ, $\gamma_{\mu\nu}$, $\gamma_\mu\gamma_5$ or γ_5 corresponding to scalar, vector, tensor, pseudovector, or pseudoscalar coupling, respectively. Sometimes it is necessary to consider not a single diagram but a set of correlating diagrams. But for simplicity we shall not take such general cases into account since an essential feature of our problem will be exhausted by this simplified treatment. As is well-known, M is often a divergent integral. In general, we do not know at all what value should be ascribed to M and thus no clue to study the problem of ambiguity. In some special cases, however, it can be proved that M should be subject to certain restrictions. They are the requirements of gauge invariance when photon fields appear among $U_i(x_i)$, divergence theorem for the vector coupling of scalar meson, and equivalence theorem between pseudoscalar and pseudovector couplings of pseudoscalar meson. (There are some cases in which the latter two theorems do not hold.) These three requirements are all alike mathematically and proved in the following formal manner. Let us assume for example that Γ_1 is γ_μ (vector type) and $U_1(x_1)=\frac{\partial}{\partial x_{1\mu}}\bar{U}_1(x_1)$. Then, integrating by parts, (1) can be rewritten as

$$M=-\int_{-\infty}^{\infty} dx_0 dx_1 \cdots dx_n U_0(x_0)\bar{U}_1(x_1) \cdots U_n(x_n)$$
$$\times S_p\Big\{\Sigma \frac{\partial}{\partial x_{1\mu}}[\bar{S}(x_0-x_1)\gamma_\mu \bar{S}(x_1-x_2)]\Gamma_2 \cdots \Gamma_n \bar{S}(x_n-x_0)\Gamma_0\Big\}$$
$$=-\int_{-\infty}^{\infty} dx_0 dx_1 \cdots dx_n U_0(x_0)\bar{U}_1(x_1) \cdots U_n(x_n)$$
$$\times S_p\Big\{\Sigma\Big[\Big(\frac{\partial}{\partial x_{1\mu}}\bar{S}(x_0-x_1)\gamma_\mu - \varkappa \bar{S}(x_0-x_1)\Big)\bar{S}(x_1-x_2)$$

---

* Other types of interaction can be treated in a similar way.

$$+\bar{S}(x_0-x_1)\Big(\gamma_\mu\frac{\partial}{\partial x_{1\mu}}+\varkappa\Big)\bar{S}(x_1-x_2)\Big]\Gamma_2\cdots\Gamma_n\bar{S}(x_n-x_0)\Gamma_0\Big\}. \tag{2}$$

In order to complete our proof we have only to use the trivial identity

$$\int_{-\infty}^{\infty}dx_0dx_1\cdots dx_nU_0(x_0)\bar{U}_1(x_1)\cdots U_n(x_n)$$

$$\times S_p\Big\{\sum\Big[\frac{\partial}{\partial x_{1\mu}}\bar{S}(x_0-x_1)\gamma_\mu-\varkappa\bar{S}(x_0-x_1)\Big]\bar{S}(x_1-x_2)\Gamma_2\cdots\Gamma_n\bar{S}(x_n-x_0)\Gamma_0\Big\}$$

$$=\int_{-\infty}^{\infty}dx_0dx_2\cdots dx_nU_0(x_0)\bar{U}_1(x_0)\cdots U_n(x_n)$$

$$\times S_p\{\sum\bar{S}(x_0-x_2)\Gamma_2\cdots\Gamma_n\bar{S}(x_n-x_0)\Gamma_0\}, \tag{3}$$

which is a direct consequence of the properties of $\bar{S}(x)$ and $S^{(1)}(x)$:

$$\Big(\gamma\frac{\partial}{\partial x}+\varkappa\Big)\bar{S}(x)=-\delta(x),$$

$$\Big(\gamma\frac{\partial}{\partial x}+\varkappa\Big)S^{(1)}(x)=0. \tag{4}$$

The case when Γ_1 is of a pseudovector type can be treated in the same way. It is thus evident that the result of actual calculation of (1) must satisfy gauge invariance, etc. if this method of calculation does not violate the relation (3). In fact, methods employed heretofore are contradictory with formal requirements and therefore with the equation (3). Let us see in the following section how the equation (3) is destroyed in the practical calculation (performed in the momentum representation).

§ 2. Calculations and discussions

The most familiar way to evaluate (1) is to express M in the momentum representation using

$$\bar{S}(x)=\frac{1}{(2\pi)^4}\int dk\,e^{ikx}\frac{i\gamma k-\varkappa}{k^2+\varkappa^2},$$

$$S^{(1)}(x)=\frac{1}{2(2\pi)^3}\int dk\,e^{ikx}(i\gamma k-\varkappa)\delta(k^2+\varkappa^2),$$

$$U(x_i)=U_ie^{ip_ix_i}. \tag{5}$$

Then (1) becomes

$$M=\pi\int dk_0dk_1\cdots dk_n\prod_{i=0}^{n}U_i\delta(k_{i-1}-k_i-p_i)\,S_p\{\prod_{j=0}^{n}(i\gamma k_j-\varkappa)\Gamma_{j+1}\}$$

$$\times\Big(\sum_{i=0}^{n}\frac{\delta(k_i^2+\varkappa^2)}{(k_0^2+\varkappa^2)\cdots(k_{i-1}^2+\varkappa^2)(k_{i+1}^2+\varkappa^2)\cdots(k_n^2+\varkappa^2)}\Big). \tag{6}$$

In this and the following formulas, it is to be understood that suffixes differing from each other by some multiples of $n+1$ are identical. As to the last bracket in (6) it is found that it is equal to

$$\int_{-1}^{1}\frac{da_1}{2}\cdots\int_{-1}^{1}\frac{da_n}{2}(-1)^n\eta_n\delta^{(n)}\left(x^2+\sum_{i=0}^{n}\xi_i k_i^2\right) \tag{7}$$ [6]

with

$$(\xi_0,\ \xi_1,\cdots,\ \xi_n)=\left(\frac{1+a_1}{2},\ \frac{1-a_1}{2}\frac{1+a_2}{2},\ \cdots\right.$$
$$\left.\cdots,\ \frac{1-a_1}{2}\frac{1-a_2}{2}\cdots\frac{1-a_{n-1}}{2}\frac{1+a_n}{2},\ \frac{1-a_1}{2}\frac{1-a_2}{2}\cdots\frac{1-a_{n-1}}{2}\frac{1-a_n}{2}\right) \tag{8}$$

and

$$\eta_n=\xi_0\prod_{i=1}^{n}\frac{\xi_i}{\frac{1-a_i^2}{4}}. \tag{9}$$

Order of terms in the bracket (8) does not matter in our discussion at all. ξ_i's satisfy the equation

$$\sum_{i=0}^{n}\xi_i=1. \tag{10}$$

For further discussions it is convenient to make the linear substitution of variables[6]

$$\left.\begin{aligned}&k_i=k+A_i\\ \text{with}\quad &A_i=\sum_{j\neq i}p_j\sum_{k=j}^{i-1}\xi_k\end{aligned}\right\}\quad i=0,\ 1,\cdots,\ n. \tag{11}$$

We can easily verify the relations

$$A_{i-1}-A_i=p_i \qquad i=0,\ 1,\ \cdots,\ n, \tag{12}$$

and

$$\sum_{i=0}^{n}\xi_i A_i=0 \tag{13}$$

employing (10) and the conservation of energy and momentum

$$\sum_{i=0}^{n}p_i=0 \tag{13'}$$

which is involved in (6). (6) is thus transformed into*

---

* When the variables of integration are transformed according to (11), it becomes dubious what variable should be considered as having a spherically symmetrical domain of integration. Various ways of integration can therefore be considered besides ours. For example, Karplus and Kroll (Phys. Rev. 77 (1950), 536) used a method which assigns non-vanishing value to surface integral that arises accompanying the change of variable None of them, however, may be satisfactory to remove ambiguities.

$$\int dk\int_{-1}^{1}\frac{da_1}{2}\cdots\int_{-1}^{1}\frac{da_n}{2}(-1)^n\eta_n\delta^{(n)}(k^2+B)\,S_p\{\prod_{j=0}^{n}(i\gamma(k+A_j)-\varkappa)\Gamma_{j+1}\} \tag{6'}$$

with

$$B=\varkappa^2+\sum_{i=0}^{n}\xi_i A_i^2 \tag{15}$$

where a numerical factor and U_i's are omitted for convenience of calculation. Formulas given below are useful in the following calculations. It is seen from (8) that*

$$(1-a_1)\frac{\partial}{\partial a_1}\xi_i=\delta_{0i}-\xi_i,\qquad i=0,\ 1,\ \cdots,\ n. \tag{16}$$

where $\delta_{0i}=1$ or 0 according as $i=0$ or otherwise. From (12) and (16) we can deduce that

$$(1-a_1)\frac{\partial}{\partial a_1}A_i=-A_0,\qquad i=0,\ 1,\ \cdots,\ n. \tag{17}$$

We are now prepared to continue the discussion at the end of § 1. The left member of (3) is expressed in the momentum representation as

$$M'=\int dk\int_{-1}^{1}\frac{da_1}{2}\cdots\int_{-1}^{1}\frac{da_n}{2}(-1)^n\eta_n\delta^{(n)}(k^2+B)\,S_p\{(\varkappa^2+(k+A_0)^2)\prod_{j=1}^{n}(i\gamma(k+A_j)-\varkappa)\Gamma_{j+1}\}$$

$$=\int dk\int_{-1}^{1}\frac{da_1}{2}\cdots\int_{-1}^{1}\frac{da_n}{2}(-1)^n\eta_n\{(k^2+B)+2kA_0+(A_0^2-\sum_{i=0}^{n}\xi_iA_i^2)\}\delta^{(n)}(k^2+B)S_p(k,A) \tag{18}$$

where

$$S_p(k,A)\equiv S_p\{\prod_{j=1}^{n}(i\gamma(k+A_j)-\varkappa)\Gamma_{j+1}\}. \tag{18'}$$

Our problem is to derive from (18) a term corresponding to the right member of (3). From (16) and (17) we get

---

* Choice of the variable $\xi_0=\frac{1+a_1}{2}$ in (8) does not destroy the generality of our argument at all. When $\xi_0=\frac{1-a_1}{2}\cdots\frac{1-a_i}{2}\frac{1+a_{i+1}}{2}$ is employed, we have only to introduce an operator

$$A_0=-\sum_{j=1}^{i}\frac{1+a_j}{\frac{1-a_1}{2}\cdots\frac{1-a_{j-1}}{2}}\frac{\partial}{\partial a_j}+\frac{1-a_{i+1}}{\frac{1-a_1}{2}\cdots\frac{1-a_i}{2}}\frac{\partial}{\partial a_{i+1}}$$

in place of $(1-a_1)\frac{\partial}{\partial a_1}$. In fact, this operator gives the relations

$$A_0\xi_i=\delta_{0i}-\xi_i$$

and

$$-\sum_{j=1}^{i}\frac{\partial}{\partial a_j}\frac{(1+a_j)\eta_n}{\frac{1-a_1}{2}\cdots\frac{1-a_{j-1}}{2}}+\frac{\partial}{\partial a_{i+1}}\frac{(1-a_{i+1})\eta_n}{\frac{1-a_1}{2}\cdots\frac{1-a_i}{2}}=-n\eta_n$$

corresponding to (16) and (22), respectively. The entire discussions can thus be carried through in the same way as is given in this paper.

$$(A_0^2-\sum_{i=0}^{n}\xi_i A_i^2)\delta^{(n)}(k^2+B)=(1-a_1)\frac{\partial}{\partial a_1}\delta^{(n-1)}(k^2+B) \tag{19}$$

where (13) is employed. Then

$$\eta_n(A_0^2-\sum_{i=0}^{n}\xi_i A_i^2)\delta^{(n)}(k^2+B)S_p(k,A)$$

$$=\frac{\partial}{\partial a_1}\{(1-a_1)\eta_n\delta^{(n-1)}(k^2+B)S_p(k,A)\}$$

$$-\delta^{(n-1)}(k^2+B)\left\{\frac{\partial}{\partial a_1}((1-a_1)\eta_n)+\eta_n(1-a_1)\frac{\partial}{\partial a_1}\right\}S_p(k,A). \tag{20}$$

The second term of (20) can be rewritten as

$$\eta_n\delta^{(n-1)}(k^2+B)\left(n+A_{0\mu}\frac{\partial}{\partial k_\mu}\right)S_p(k,A) \tag{21}$$

since

$$\frac{\partial}{\partial a_1}((1-a_1)\eta_n)=-n\eta_n,$$

$$(1-a_1)\frac{\partial}{\partial a_1}(i\gamma(k+A_s)-\varkappa)=-i\gamma A_0=-A_{0\mu}\frac{\partial}{\partial k_\mu}(i\gamma(k+A_s)-\varkappa). \tag{22}$$

(18) thus becomes

$$M'=\int dk\int_{-1}^{1}\frac{da_1}{2}\cdots\int_{-1}^{1}\frac{da_n}{2}(-1)^n\eta_n\Big\{(k^2+B)\delta^{(n)}(k^2+B)$$

$$+\left(n+A_{0\mu}\frac{\partial}{\partial k_\mu}\right)\delta^{(n-1)}(k^2+B)\Big\}S_p(k,A)$$

$$+\int dk\int_{-1}^{1}\frac{da_1}{2}\cdots\int_{-1}^{1}\frac{da_n}{2}(-1)^n\frac{\partial}{\partial a_1}\{(1-a_1)\eta_n\delta^{(n-1)}(k^2+B)S_p(k,A)\}. \tag{23}$$

Integrating the last term with respect to a_1, one finds

$$\int dk\int_{-1}^{1}\frac{da_2}{2}\cdots\int_{-1}^{1}\frac{da_n}{2}\frac{(-1)^n}{2}\{(1-a_1)\eta_n\delta^{(n-1)}(k^2+B)S_p(k,A)\}\Big|_{a_1=-1}^{a_1=1}$$

$$=\int dk\int_{-1}^{1}\frac{da_2}{2}\cdots\int_{-1}^{1}\frac{da_n}{2}(-1)^{n-1}\eta_n'\delta^{(n-1)}(k^2+\varkappa^2+\sum_{i=1}^{n}\xi_i'A_i'^2)S_p(k,A') \tag{24}$$

where ξ_i', A_i', and η_n' mean values of ξ_i, A_i, and η_n for $a_1=-1$ respectively. This is just the right hand side of (3) in the momentum representation. It is the first term of (23) that destroys the validity of equation (3). We can thus conclude that the first term of (23) has to vanish in order that the equation (3) holds in the actual calculation employing the momentum variables.

For this purpose, it is sufficient that the equation

$$\int dk \Big\{ k^l((k^2+B)\delta^{(n)}(k^2+B)+n\delta^{(n-1)}(k^2+B))+A_{0\mu}\frac{\partial}{\partial k_\mu}(k^l\delta^{(n-1)}(k^2+B))\Big\}=0 \quad (25)$$

holds for $l=0, 1, \cdots, n$, where k^l is a symbolical representation of an arbitrary product of l factors k_ν. When this integral is absolutely convergent, (25) is satisfied identically and moreover can be divided into two conditions

$$\int dk\, k^l\{(k^2+B)\delta^{(n)}(k^2+B)+n\delta^{(n-1)}(k^2+B)\}=0 \qquad (26.\mathrm{I})$$

for even l not exceeding n and

$$\int dk \frac{\partial}{\partial k_\mu}\{k^l\delta^{(n-1)}(k^2+B)\}=0 \qquad (26.\mathrm{II})$$

for odd l not exceeding n. In the following discussion, we shall obey the convention that integrals of odd functions are always zero when integrated over all momentum space even though it is really divergent. Then (26. I) and (26. II) can be regarded as perfect substitutions for the condition (25). Furthermore it is easily found that when n is even the condition (26. I) for $l=n$ is superfluous for the actual proof of theorems since a condition of the same form always appears with opposite sign from the other part of (2) which we did not write explicitly to simplify our discussion. Now, since

$$\int dk\delta^{(n)}(k^2+B)=\textit{convergent for} \quad 2(n+1)>4$$

or more generally

$$\int dk\, k^l\delta^{(n)}(k^2+B)=\textit{convergent for} \quad 2(n+1)>l+4, \qquad (27)$$

the left hand side of (26. I) converges if

$$2n>l+4.$$

This relation surely holds when $n\geqq 4$, since, for $n=4$, l can not exceed 2. The condition for the convergence of the integral of (26. II) is also $n\geqq 4$. It is thus obvious that conditions (26. I) and (26. II) are trivial for $n\geqq 4$. For $n\leqq 3$ (except for the cases $n=3$, $l=0, 1$), these integrals do not converge in general and therefore we have to establish them as real conditions which have to be imposed so that the method of calculation is free from ambiguities. (It is a matter of course that even for $n\leqq 3$ we may come across with such cases that (26. I) and (26. II) become trivial on account of special circumstance.) We shall now discuss the remaining conditions in detail.

For $n=1$, the conditions are

$$\int dk\{(k^2+B)\delta'(k^2+B)+\delta(k^2+B)\}=0 \qquad (28.\mathrm{I})$$

and

$$\int dk \frac{\partial}{\partial k_\mu}\{k_\nu \delta(k^2+B)\}=0. \tag{29}$$

It is to be noted tnat (29) is equivalent to

$$\int dk\{k^2\delta'(k^2+B)+2\delta(k^2+B)\}=0 \tag{28. II}$$

since this is obtained by contractio.ı of the equation

$$\int dk \frac{\partial}{\partial k_\mu}\{k_\nu \delta(k^2+B)\}=\delta_{\mu\nu}\int dk\delta(k^2+B)+2\int dk k_\mu k_\nu \delta'(k^2+B).$$

In the same way, conditions for $n=2$ are

$$\int dk\{(k^2+B)\delta''(k^2+B)+2\delta'(k^2+B)\}=0, \tag{30. I}$$

$$\int dk\{k^2\delta''(k^2+B)+2\delta'(k^2+B)\}=0, \tag{30. II}$$

and conditions for $n=3$ are

$$\int dk\, k^2\{(k^2+B)\delta'''(k^2+B)+3\delta''(k^2+B)\}=0, \tag{31. I}$$

$$\int dk\, k^2\{k^2\delta'''(k^2+B)+3\delta''(k^2+B)\}=0. \tag{31. II}$$

Six conditions obtained above are obviously classified into two groups. Group I consists of (28. I), (30. I), and (31. I). They can all be derived differentiating the next formal identity

$$\int dk(k^2+B)^a\delta(k^2+B)=0, \qquad a=1 \text{ or } 2,$$

several times with respect to B. The other set II are composed of (28. II), (30. II), and (31. II). As is easily seen from (26. II), these conditions merely mean that we must always drop surface integrals when we perform partial integration. In order to remove all ambiguities we need therefore two sets of conditions which are mutually independent and contradictory to each other. It is to be stressed that a single set of conditions is not sufficient.*

In performing actual calculation, however, it may be more convenient to use only one of alternative definitions of indefinite integral throughout than to make use of both conditions at the same time. Then, in order to find unambiguous results, we have to employ the auxiliary conditions

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* In a recent work concerning infinite integrals (Phys. Rev. in press), Peaselee suggested that only the condition II was to be adopted for the treatment of divergent expressions. In our opinion, his method also is not satisfactory for the present field theories which suffer from the difficulty of divergence. The authors wish to thank Prof. Peaselee for the opportunity of seeing his manuscript before publication.

$$\int dk(k^2+2B)\delta'(k^2+B)=0 \quad \text{for} \quad n=1, \tag{32}$$

$$\int dkB\delta''(k^2+B)=0 \quad \text{for} \quad n=2, \tag{33}$$

$$\int dkBk^2\delta'''(k^2+B)=0 \quad \text{for} \quad n=3, \tag{34}$$

i.e., differences of two sets of conditions, in addition to the adopted set of conditions. (32) and (33) are nothing else but the conditions which Katayama discovered in his analysis of the problems of ambiguities. He proved that the condition (32) is sufficient to remove every ambiguous term in the case $n=1$, while he tested the effectiveness of (33) only for restricted range of examples belonging to $n=2$. We have confirmed here in a general manner that (33) is just sufficient to get rid of all possible ambiguities which appear when $n=2$. (33) and (34) are rather curious conditions since they are actually convergent integrals and have definite values π and -2π respectively. Use of such mathematically incorrect conditions seems to be inevitable so long as the problem of divergence remains unsolved.

The authors wish to thank Messrs. G. Takeda and K. Aidzu for their valuable discussions.

Note added in the proof

Attempts to find weaker conditions than those given in this paper are made by Koba et. al. in connection with the fourth order calculation of the production of π-mesons by X-rays (not yet published).

References

1) G. Wentzel, Phys. Rev. **74** (1948), 1070.
H. Fukuda and Y. Miyamoto, Prog. Theor. Phys. **4** (1949), 347.
H. Fukuda, S. Hayakawa and Y. Miyamoto, Prog. Theor. Phys. **5** (1950), 283.
2) W. Pauli and F. Villars, Rev. Mod. Phys. **21** (1949), 434.
3) H. Fukuda, Y. Miyamoto, T. Miyazima, S. Tomonaga, S. Ozaki, S. Ôneda and S. Sasaki, Prog. Theor. Phys. **4** (1949), 477.
4) H. Umezawa and R. Kawabe, Prog. Theor. Phys. **4** (1949), 423, 443.
D. Feldman, Phys. Rev. **76** (1949), 1369.
R. Jost and J. Rayski, Helv. Phys. Acta **22** (1949), 458.
5) Y. Katayama, Prog. Theor. Phys. **5** (1950), 272.
6) H. Fukuda, S. Hayakawa and Y. Miyamoto, Prog. Theor. Phys. **5** (1950), 352.

Progress of Theoretical Physics, Vol. 5, No. 6, November~December, 1950

Notes on Dirac's New Quantization Method in the Field Theory

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(Received August 11, 1950)

§ 1. Introduction and summary

Although many attempts have been made so far by various theoretical physicists for removing the well-known divergency-difficulties inherent to the quantum theory of wave fields, we unfortunately have not yet succeeded to derive a suitable formulation of the field theory free from the mentioned difficulties in a satisfactory manner. The self-consistent subtraction theory proposed by Bethe, Schwinger and Tomonaga[1] has been developed with brilliant success, in the field of quantum electrodynamics, in accordance with the beautiful experiments of Lamb and Retherford[2], and Rabi and his coworkers[3], but it cannot claim its universal validity for all kinds of wave fields accessible at present since, for example, the renormalization of an electric charge closely associated with the vacuum polarization has been shown to be entirely invalid in the vector meson theory[4]. Recently, Dirac's new method[5] of field quantization associated with the introduction of negative energy photon has been proposed with some success to remove the divergency of mass term in one electron theory. Although his theory has entirely failed to eliminate the mentioned divergency in the hole theory together with the inclusion of another difficulty concerning the physical interpretation of negative probability, it seems to provide us a provisional but attractive footing for the future development of the field theory since it is based on the quite natural generalization of the metric in Hilbert space upon which the mathematical scheme of the ordinary quantum mechanics has been formulated in a consistent way. In fact, if Pauli's qualitative consideration of Dirac's theory mentioned above were actually correct, the self-energy, to any degree of order, of an electron would be brought to vanish in one electron problem[6], which fact seems to draw much attention in view of the present situation that the provisional theories proposed so far are almost restricted merely to the second order self-energy. After all, since the merit or demerit of Dirac's new method of field quantization should be eventually criticized only through its application to the various kinds of physical phenomena in a close connection with the experimental results, we have here worked out the fourth order self-energy of a free electron and the Lamb-shift, i.e., the second order radiative reaction of a bound electron in hydrogen atom in order to provide a useful data for judging the physical

contents of Dirac's new method of field-quantization.

According to our computation the fourth order self-energy of a free electron has been found to diverge, in contradistinction to Pauli's qualitative expectation[6], owing to the appearance of a transcendental function in the integrand involved in the photon-momenta integrals. As for the second order self-energy, however, the integrand of a photon-momentum in the intermediate states has always become of a rational function; its even part has been made to vanish by the use of λ-limiting process while its odd part by introducing negative energy photons respectively. On the other hand, since the fourth order self-energy involves two photons simultaneously in the intermediate states, the mutual correlations between both photon-mementum vectors are shown to be responsible for the appearance of the irrational function in the integrand, which makes Dirac's new method entirely invalid for the convergency problem.

The Lamb shift of a hydrogen level has been worked out relativistically or non-relativistically by many authors[7]. In the relativistic derivation the renormalization of an electronic mass has been adopted to avoid the well-known divergency in mass term, while in the non-relativistic one, in addition, the artificial cutting-off of a photon momentum in the intermediate state has been introduced by Bethe[5]. In order to see whether Dirac's new procedure is capable of introducing the automatic cutting-off instead of Bethe's artificial one or not, we have worked out the Lamb shift of a hydrogen level in accordance with Dirac's new method of field quantization, then the electronic motion being treated non-relativistically as in the case of Bethe: which computation seems to provide us some information about the limit of the usefulness of Dirac's new procedure, though qualitative. The computed Lamb-shift has been found to vanish in a drastic disagreement with the experimental observations.

Thus, our results reveal that Dirac's cutting-off involved in his new field quantization is too severe for the second order reaction of the electromagnetic interaction with a free or bound electron, while, for the fourth order one, it can hardly suppress the corresponding divergence in the one electron treatment. It, therefore, seems impossible to develop the present form of his theory without its essential modification.

§ 2. The fourth order self-energy of a free electron

(1) *Derivation of the fourth order self-energy*

Dirac's wave equation of an electron interacting with an electromagnetic field may be written, as usual, by

$$p_0\Psi=\{c\boldsymbol{\alpha p}+mc^2\beta+\Omega\}\Psi, \tag{1}$$

where

$$\Omega=\left(\frac{2\pi\hbar c}{V}\right)^{1/2}c\sum_{\boldsymbol{k}}k_0^{-\frac{1}{2}}\alpha\left[\frac{1}{\sqrt{2}}(\boldsymbol{U}_+^T(\boldsymbol{k})+\boldsymbol{U}_-^{T*}(\boldsymbol{k}))e^{i(\boldsymbol{kx}-k_0x_0)}+\right.$$

$$+\frac{1}{\sqrt{2}}(U_+^{T*}(\boldsymbol{k})+U_-^T(\boldsymbol{k}))e^{-i(\boldsymbol{kx}-k_0x_0)}\Big], \tag{2}$$

$p_0=i\hbar\frac{\partial}{\partial t}=i\hbar c\frac{\partial}{\partial x_0}$, $\boldsymbol{p}=-i\hbar\text{grad}$, e = electronic charge, m=electronic rest mass, c=light velocity, $\hbar$=Planck's constant divided by 2π, $\boldsymbol{\alpha}$, β=Dirac's spin operators, $\boldsymbol{k}$, k_0=momentum and energy of a photon and V=fundamental volume under consideration.

Further, $U_+^T(\boldsymbol{k})$ and $U_+^{T*}(\boldsymbol{k})$ represent the emission and absorption operators of a positive energy photon $\boldsymbol{k}$, and $U_-^T(\boldsymbol{k})$ and $U_-^{T*}(\boldsymbol{k})$ those of a negative energy photon $-\boldsymbol{k}$. According to Dirac's new method of field quantization, the mentioned operators have to satisfy the following commutation relations[6]:

$$[U^T_{+,i}(\boldsymbol{k}),\ U^{T*}_{+,j}(\boldsymbol{k})]=-[U^T_{-,i}(\boldsymbol{k}),\ U^T_{-,j}{}^*(\boldsymbol{k})]=\left(\delta_{ij}-\frac{k_ik_j}{k_0^2}\right)\cos(k_0\lambda_0-\boldsymbol{k\lambda}), \tag{3}$$

where $(\lambda_0, \boldsymbol{\lambda})$ denotes a time-like vector which is to be put equal to zero in the final result.

In the sense of the perturbation theory, we write

$$\Psi=\Psi_0+\Psi_1+\Psi_2+\cdots\cdots, \tag{4}$$

where the term Ψ_n is of the order e^n and obtain, as usual,

$$\left.\begin{aligned}
&(p_0-c\boldsymbol{\alpha p}-mc^2\beta)\Psi_0=0, && (5.1)\\
&(p_0-c\boldsymbol{\alpha p}-mc^2\beta)\Psi_1=\Omega\Psi_0, && (5.2)\\
&(p_0-c\boldsymbol{\alpha p}-mc^2\beta)\Psi_2=\Omega\Psi_1, && (5.3)\\
&(p_0-c\boldsymbol{\alpha p}-mc^2\beta)\Psi_3=\Omega\Psi_2, && (5.4)\\
&(p_0-c\boldsymbol{\alpha p}-mc^2\beta)\Psi_4=\Omega\Psi_3, && (5.5)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \tag{5}$$

corresponding to each degree of the approximations.

We start with the state of a single electron in vacuum, i.e., without photon $(N_+(\boldsymbol{k})=N_-(\boldsymbol{k})=0)$, in which $N_+(\boldsymbol{k})$ and $N_-(\boldsymbol{k})$ represent photon numbers of positive and negative energies respectively. Writing the corresponding photon eigenfunction by w_0 and the electron eigenfunction by u_0, we have, for the solution of (5.1),

$$\Psi_0=u_0w_0, \tag{6}$$

where

$$u_0=\boldsymbol{a}\exp\left\{\frac{i}{\hbar}(\boldsymbol{qx}-c^{-1}q_0x_0)\right\} \tag{6$'$}$$

and $\boldsymbol{q}$, q_0 satisfy the energy equation

$$q_0^2-c^2\boldsymbol{q}^2-m^2c^4=0, \tag{6$''$}$$

$\boldsymbol{a}$ being the well-known spinor amplitude.

Ψ_1 contains only states where one photon is present, while Ψ_2 contains states with no photon and with two photons. Writing the eigenfunctions of these states $\Psi_{2,0}$ and $\Psi_{2,2}$ respectively, we can put

$$\Psi_2=\Psi_{20}+\Psi_{22}. \tag{7}$$

Similarly, we have

$$\Psi_3=\Psi_{31}+\Psi_{33}, \tag{8}$$

$$\Psi_4=\Psi_{40}+\Psi_{42}+\Psi_{44}, \tag{9}$$

according to a simple consideration.

For the computation of the fourth order self-energy, we require Ψ_{40} only which is connected with the initial state Ψ_0 through the intermediate states as shown in the following diagram.

initial state intermediate state final state

$$\begin{tikzcd}
 & & \Psi_{20} \arrow[dr, dashed] & & \\
\Psi_0 \arrow[r] & \Psi_1 \arrow[ur, dashed] \arrow[dr, dashed] & & \Psi_{31} \arrow[r] & \Psi_{40} \\
 & & \Psi_{22} \arrow[ur, dashed] & &
\end{tikzcd}$$

The intermediate states in the upper branch of the above diagram is easily found to make no contribution to the fourth order self energy in view of the second order self energy[6] being equal to zero, which situation is seen to save greatly our process of manipulation.

From (5.2), we get

$$\Psi_1=\left(\frac{2\pi\hbar c}{V}\right)^{1/2} e\sum_{\mathbf{k}} k_0^{-\frac{1}{2}}\frac{1}{\sqrt{2}}\Bigg[\frac{(q_0+c\hbar k_0)+c\boldsymbol{\alpha}(\boldsymbol{q}+\hbar\boldsymbol{k})+mc^2\beta}{(q_0+c\hbar k_0)^2-c^2(\boldsymbol{q}+\hbar\boldsymbol{k})^2-m^2c^4}(\boldsymbol{\alpha}\cdot\boldsymbol{U}_+^T(\boldsymbol{k}))\cdot$$
$$\cdot e^{\frac{i}{\hbar}\{(\boldsymbol{q}+\hbar\boldsymbol{k})\boldsymbol{x}-(c^{-1}q_0+\hbar k_0)x_0\}}$$
$$+\frac{(q_0-c\hbar k_0)+c\boldsymbol{\alpha}(\boldsymbol{q}-\hbar\boldsymbol{k})+mc^2\beta}{(q_0-c\hbar k_0)^2-c^2(\boldsymbol{q}-\hbar\boldsymbol{k})^2-m^2c^4}(\boldsymbol{\alpha}\cdot\boldsymbol{U}_-^T(\boldsymbol{k}))\cdot$$
$$\cdot e^{\frac{i}{\hbar}\{(\boldsymbol{q}-\hbar\boldsymbol{k})\boldsymbol{x}-(c^{-1}q_0-\hbar k_0)x_0\}}\Bigg]\Psi_0, \tag{10}$$

using the relations

$$\boldsymbol{U}_+^{T*}(\boldsymbol{k})\Psi_0=\boldsymbol{U}_-^{T*}(\boldsymbol{k})\Psi_0=0 \tag{11}$$

and

$$(p_0-c\boldsymbol{\alpha}\boldsymbol{p}-mc^2\beta)^{-1}=\frac{p_0+c\boldsymbol{\alpha}\boldsymbol{p}+mc^2\beta}{p_0^2-c^2\boldsymbol{p}^2-m^2c^4}. \tag{12}$$

Similarly, from (5.3) and (5.4), we shall be able to obtain the corresponding solutions Ψ_{22} and Ψ_{31} respectively, which shall be omitted here owing to the rather lengthy expressions.

Finally, substituting the obtained result for Ψ_{31} into (5.5), we have the following equation to be satisfied by Ψ_{40}:

$$
\begin{aligned}
(p_0 - c\boldsymbol{\alpha p} - mc^2\beta)\Psi_{40} = & \left(\frac{\pi\hbar c}{l}\right)^2 c^4 \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k}'',\boldsymbol{k}'''} (k_0 k_0' k_0'' k_0''')^{-\frac{1}{2}} \cdot \\
& \cdot\Big[(\boldsymbol{\alpha} U_+^{T*}(\boldsymbol{k}''')) \frac{(q_0 + c\hbar(k_0 + k_0' - k_0''))}{2c\hbar(q_0(k_0+k_0'-k_0'') - c\boldsymbol{q}(\boldsymbol{k}+\boldsymbol{k}'-\boldsymbol{k}'')} \\
& \qquad \frac{+c\boldsymbol{\alpha}(\boldsymbol{q}+\hbar(\boldsymbol{k}+\boldsymbol{k}'-\boldsymbol{k}'')) + mc^2\beta}{+c\hbar(k_0k_0' - k_0k_0'' - k_0'k_0'' - \boldsymbol{kk}' + \boldsymbol{kk}'' + \boldsymbol{k}'\boldsymbol{k}'')} (\boldsymbol{\alpha U}_+^{T*}(\boldsymbol{k}''))\cdot \\
& \cdot \frac{(q_0 + c\hbar(k_0 + k_0')) + c\boldsymbol{\alpha}(\boldsymbol{q} + \hbar(\boldsymbol{k}+\boldsymbol{k}')) + mc^2\beta}{2c\hbar(q_0(k_0+k_0') - c\boldsymbol{q}(\boldsymbol{k}+\boldsymbol{k}') + c\hbar(k_0k_0' - \boldsymbol{kk}'))} (\boldsymbol{\alpha U}_+^T(\boldsymbol{k}'))\cdot \\
& \quad \cdot \frac{(q_0 + c\hbar k_0) + c\boldsymbol{\alpha}(\boldsymbol{q}+\hbar\boldsymbol{k}) + mc^2\beta}{2c\hbar(q_0k_0 - c\boldsymbol{qk})} (\boldsymbol{\alpha U}_+^T(\boldsymbol{k}))\cdot \\
& \cdot e^{\frac{i}{\hbar}\{(\boldsymbol{q}+\hbar(\boldsymbol{k}+\boldsymbol{k}'-\boldsymbol{k}''-\boldsymbol{k}'''))\boldsymbol{x} - (c^{-1}q_0 + \hbar(k_0+k_0'-k_0''-k_0'''))x_0\}} \\
+ & \{(k_0', \boldsymbol{k}'),\ \boldsymbol{U}_+^T(\boldsymbol{k}');\ (k_0''', \boldsymbol{k}'''),\ \boldsymbol{U}_+^{T*}(\boldsymbol{k}''') \\
& \qquad \to (-k_0', -\boldsymbol{k}),\ \boldsymbol{U}_-^T(\boldsymbol{k}');\ (-k_0''', -\boldsymbol{k}'''),\ \boldsymbol{U}_-^{T*}(\boldsymbol{k}''')\} \\
+ & \{(k_0', \boldsymbol{k}'),\ \boldsymbol{U}_+^T(\boldsymbol{k}');\ (k_0'', \boldsymbol{k}''),\ \boldsymbol{U}_+^{T*}(\boldsymbol{k}'') \\
& \qquad \to (-k_0', -\boldsymbol{k}'),\ \boldsymbol{U}_-^T(\boldsymbol{k}');\ (-k_0'', -\boldsymbol{k}''),\ \boldsymbol{U}_-^{T*}(\boldsymbol{k}'')\} \\
+ & \{(k_0, \boldsymbol{k}),\ \boldsymbol{U}_+^T(\boldsymbol{k});\ (k_0''', \boldsymbol{k}'''),\ \boldsymbol{U}_+^{T*}(\boldsymbol{k}''') \\
& \qquad \to (-k_0, -\boldsymbol{k}),\ \boldsymbol{U}_-^T(\boldsymbol{k});\ (-k_0''', -\boldsymbol{k}'''),\ \boldsymbol{U}_-^{T*}(\boldsymbol{k}''')\} \\
+ & \{(k_0, \boldsymbol{k}),\ \boldsymbol{U}_+^T(\boldsymbol{k});\ (k_0'', \boldsymbol{k}''),\ \boldsymbol{U}_+^{T*}(\boldsymbol{k}'') \\
& \qquad \to (-k_0, -\boldsymbol{k}),\ \boldsymbol{U}_-^T(\boldsymbol{k});\ (-k_0'', -\boldsymbol{k}''),\ \boldsymbol{U}_-^{T*}(\boldsymbol{k}'')\} \\
+ & \{(k_0, \boldsymbol{k}),\ \boldsymbol{U}_+^T(\boldsymbol{k});\ (k_0', \boldsymbol{k}'),\ \boldsymbol{U}_+^T(\boldsymbol{k}');\ (k_0'', \boldsymbol{k}''),\ \boldsymbol{U}_+^{T*}(\boldsymbol{k}'');\ (k_0''', \boldsymbol{k}'''),\ \boldsymbol{U}_+^{T*}(\boldsymbol{k}''') \\
& \qquad \to (-k_0, -\boldsymbol{k}),\ \boldsymbol{U}_-^T(\boldsymbol{k});\ (-k_0', -\boldsymbol{k}'),\ \boldsymbol{U}_-^T(\boldsymbol{k}'); \\
& \quad (-k_0'', -\boldsymbol{k}''),\ \boldsymbol{U}_-^{T*}(\boldsymbol{k}'');\ (-k_0''', -\boldsymbol{k}'''),\ \boldsymbol{U}_-^{T*}(\boldsymbol{k}''')\}\Big] aw_0. \qquad (13)
\end{aligned}
$$

By the use of the relation

$$
\begin{aligned}
& U_{+,i}^{T*}(\boldsymbol{k}''') U_{+,j}^{T*}(\boldsymbol{k}'') U_{+,l}^T(\boldsymbol{k}') U_{+,m}^T(\boldsymbol{k}) \Psi_0 \\
= & \left[\left(\delta_{il} - \frac{k_i' k_l'}{k_0'^2}\right)\left(\delta_{jm} - \frac{k_j k_m}{k_0^2}\right) \delta_{\boldsymbol{k}'\boldsymbol{k}'''} \delta_{\boldsymbol{k}\boldsymbol{k}''} + \left(\delta_{jl} - \frac{k_j' k_l'}{k_0'^2}\right)\left(\delta_{im} - \frac{k_i k_m}{k_0^2}\right) \delta_{\boldsymbol{k}'\boldsymbol{k}''} \delta_{\boldsymbol{k}\boldsymbol{k}'''} \right] \cdot \\
& \cdot \cos(\lambda_0 k_0 - \boldsymbol{\lambda k}) \cos(\lambda_0 k_0' - \boldsymbol{\lambda k}') \Psi_0,
\end{aligned}
$$

the right hand side of the equation (13) may be simplified to

$$(p_0 - c\boldsymbol{\alpha p} - mc^2\beta)u_{40} = \left(\frac{\pi\hbar c}{V}\right)^2 e^4 (2c\hbar)^{-3} \sum_{\boldsymbol{kk'}} (k_0 k_0')^{-1} \sum_{ijlm} \cdot$$

$$\cdot\Big[\alpha_i \frac{(q_0 + c\hbar k_0') + c\boldsymbol{\alpha}(\boldsymbol{q}+\hbar\boldsymbol{k'}) + mc^2\beta}{q_0 k_0' - c\boldsymbol{qk'}} \alpha_j \frac{(q_0 + c\hbar(k_0+k_0')) + c\boldsymbol{\alpha}(\boldsymbol{q}+\hbar(\boldsymbol{k}+\boldsymbol{k'})) + mc^2\beta}{q_0(k_0+k_0') - c\boldsymbol{q}(\boldsymbol{k}+\boldsymbol{k'}) + c\hbar(k_0k_0' - \boldsymbol{kk'})} \cdot$$

$$\cdot\,\alpha_l \frac{(q_0 + c\hbar k_0) + c\boldsymbol{\alpha}(\boldsymbol{q}+\hbar\boldsymbol{k}) + mc^2\beta}{q_0k_0 - c\boldsymbol{qk}} \alpha_m \left(\delta_{il} - \frac{k_i' k_l'}{k_0'^2}\right)\left(\delta_{jm} - \frac{k_j k_m}{k_0^2}\right)$$

$$+\alpha_i \frac{(q_0 + c\hbar k_0) + c\boldsymbol{\alpha}(\boldsymbol{q}+\hbar\boldsymbol{k}) + mc^2\beta}{q_0k_0 - c\boldsymbol{qk}} \alpha_j \frac{(q_0 + c\hbar(k_0+k_0')) + c\boldsymbol{\alpha}(\boldsymbol{q}+\hbar(\boldsymbol{k}+\boldsymbol{k'})) + mc^2\beta}{q_0(k_0+k_0') - c\boldsymbol{q}(\boldsymbol{k}+\boldsymbol{k'}) + c\hbar(k_0k_0' - \boldsymbol{kk'})} \cdot$$

$$\cdot\,\alpha_l \frac{(q_0 + c\hbar k_0) + c\boldsymbol{\alpha}(\boldsymbol{q}+\hbar\boldsymbol{k}) + mc^2\beta}{q_0k_0 - c\boldsymbol{qk}} \alpha_m \left(\delta_{jl} - \frac{k_j' k_l'}{k_0'^2}\right)\left(\delta_{im} - \frac{k_i k_m}{k_0^2}\right)$$

$$-\{(k_0', \boldsymbol{k'}) \to (-k_0', -\boldsymbol{k'})\} - \{(k_0, \boldsymbol{k}) \to (-k_0, -\boldsymbol{k})\}$$

$$+\{(k_0, \boldsymbol{k}), (k_0', \boldsymbol{k'}) \to (-k_0, -\boldsymbol{k}), (-k_0', -\boldsymbol{k'})\}\Big] \cos(\lambda_0 k_0' - \boldsymbol{\lambda k'})\cos(\lambda_0 k_0 - \boldsymbol{\lambda k}) u_0, \tag{14}$$

dropping the photon wave function $w_0(\Psi_{40} = u_{40}w_0)$ on both sides of (13).

We, for brevity, shall take the case of an electron at rest, i.e., $\boldsymbol{q}=0$, $q_0 = mc^2$, and further, make some rearrangements in the expression (14) by the use of the commutation relations of $\boldsymbol{\alpha}$ matrix and the associated ones, i.e., $\alpha_1\alpha_2 = i\alpha_3$ etc. Applying to the equation manipulated as above the operator $(p_0 + c\boldsymbol{\alpha p} + mc^2\beta)$ from the left side, we obtain

$$(p_0^2 - c^2\boldsymbol{p}^2 - m^2c^4)u_{40} = \frac{c\hbar e^4}{64\pi^4 q_0} \iint [(k_0k_0')^{-1}\{(q_0 + c\hbar(k_0 + k_0'))(\cos^2\theta - 1)$$

$$+2(q_0 + c\hbar k_0(1-\cos^2\theta) + c\hbar k_0'(1-\cos\theta))\}(q_0(k_0+k_0') + c\hbar k_0k_0'(1-\cos\theta))^{-1}$$

$$+(k_0' \to -k_0') + (k_0 \to -k_0) + (k_0, k_0' \to -k_0, -k_0')] \cdot$$

$$\cdot\cos(k_0\lambda_0 - \boldsymbol{\lambda k})\cos(\lambda_0 k_0' - \boldsymbol{\lambda k'}) d\boldsymbol{k} d\boldsymbol{k'}\, u_0, \tag{15}$$

inserting the integration with respect to $\boldsymbol{k}$ instead of the summation, i.e.,

$$\frac{1}{V}\sum_{\mathbf{k}} \cdots\cdots = \frac{1}{(2\pi)^3}\int d\boldsymbol{k} \cdots\cdots,$$

where θ denotes the angle between $\boldsymbol{k}$ and $\boldsymbol{k'}$ of photon momenta in the intermediate states.

The equation (15) may easily be written, to the approximation of order e^4, as

$$(p_0^2 - c^2\boldsymbol{p}^2 - m^2c^4 - \Delta(m^2c^4))(u_0 + u_{20} + u_{40}) = 0, \tag{16}$$

where $\Delta(m^2c^4)$ means simply the correction of the square of the rest energy of an electron due to the fourth order radiativé reaction, the second order mass correction being zero as already shown by Dirac and Pauli[8].

Namely, it follows

$$\Delta(m^2c^4)=\frac{e^4}{8\pi^2}\int_0^\infty dk_0\int_0^\infty dk_0'\cos(\lambda_0k_0)\cos(\lambda_0k_0')\cdot$$

$$\cdot\Bigg[\Bigg\{\frac{k_0+k_0'}{c\hbar k_0k_0'}(q_0+c\hbar(k_0+k_0'))\left(2+\frac{q_0(k_0+k_0')}{c\hbar k_0k_0'}\right)$$

$$-2\left(\frac{k_0}{k_0'}+\frac{(k_0+k_0')^2}{k_0k_0'}+\frac{q_0}{c\hbar k_0k_0'^2}(k_0+k_0')^2\right)\Bigg\}\log\left(1+\frac{2c\hbar k_0k_0'}{q_0(k_0+k_0')}\right)$$

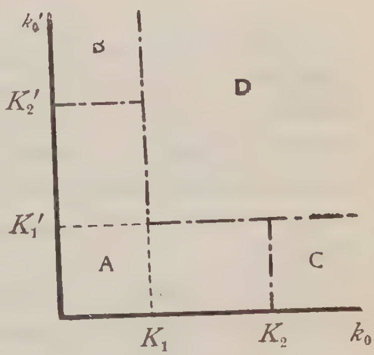
$$+\{k_0,\to -k_0\}+\{k_0'\to -k_0'\}+\{k_0,k_0'\to -k_0,-k_0'\}\Bigg], \tag{17}$$

in which the integrations with respect to the angular part have already been carried out and, further, the spatial components of four vector λ have been put equal to zero.

In contradistinction to Pauli's expectation, the integrand in (17) is seen not to become of a rational function of k and k' but involves the logarithmic function, on account of which Pauli's discussion of the convergency of the integral appearing in the energy of the higher order has been confirmed to be invalid.

(2) *Discussion of the fourth order self-energy* (17)

Owing to the double integral appearing in (17), it will be a complicated problem to investigate in detail whether the fourth order self-energy really diverges or not. We, therefore, shall give a rather qualitative discussion about the convergency-problem, showing only the outline of the rather lengthy and laborious computations on which the mentioned discussion is based, and shall reserve the detailed discussion in later.



$K_1, K_2, K_1', K_2' \gg q_0/c\hbar$
$K_1'/K_1 \sim 1$
$K_2/K_1,\ K_2'/K_1,\ K_2/K_1',\ K_2'/K_1' \gg 1$

Fig. 1

First, we have divided the integration region of k_0 and k_0' into the four parts of A, B, C and D, which are graphically represented in Fig. 1.

In the partial region denoted by A, B and C, the process of manipulations has to be carefully performed in the neighbourhood of the poles of the integrand, on account of which the integral in (17) is easily shown to remain finite even in the limit of $\lambda_0\to 0$. In the partial region of D, in which $K_1, K_1'\gg q_0/c\hbar$, we obtain after the approximate cancellation of some terms involved in the integrand.

$$\Delta(m^2c^4)\approx\frac{e^4}{8\pi^2}\int_{K_1}^\infty dk_0\int_{K_1'}^\infty dk_0'\cos(\lambda_0k_0)\cos(\lambda_0k_0')f(k_0,k_0'), \tag{18}$$

$$f(k_0, k_0') = 2\left[\left\{\frac{q_0(k_0+k_0')}{c\hbar k_0 k_0'}\right\}^2 - 2\frac{k_0}{k_0'}\right]\log\left|\frac{2c\hbar k_0 k_0'}{q_0(k_0+k_0')}\right|$$
$$+2\left[\left\{\frac{q_0(k_0-k_0')}{c\hbar k_0 k_0'}\right\}^2 + 2\frac{k_0}{k_0'}\right]\log\left|\frac{2c\hbar k_0 k_0'}{q_0(k_0-k_0')}\right| \tag{19}$$

$$= -2\left[\left\{\frac{q_0}{c\hbar}(k_0+k_0')\right\}^2 \log\left|\frac{q_0}{c\hbar}(k_0+k_0')\right| + \left\{\frac{q_0}{c\hbar}(k_0-k_0')\right\}^2 \log\left|\frac{q_0}{c\hbar}(k_0-k_0')\right|\right]$$
$$+4\frac{k_0}{k_0'}\log\left|\frac{k_0+k_0'}{k_0-k_0'}\right|. \tag{19'}$$

After dropping the bounded integrals involved in (18), we get finally

$$\Delta m^2 c^4 \approx \frac{e^4}{2\pi^2}\int_{K_1}^{\infty} dk_0 \int_{K_1'}^{\infty} dk_0' \cos(\lambda_0 k_0)\cos(\lambda_0 k_0')\frac{k_0}{k_0'}\log\left|\frac{k_0+k_0'}{k_0-k_0'}\right|$$
$$= \frac{e^4}{4\pi^2}\int_{K_1}^{\infty} dk_0 \int_{K_1'}^{\infty} dk_0' \{\cos\lambda_0(k_0+k_0') + \cos\lambda_0(k_0-k_0')\}\frac{k_0}{k_0'}\log\left|\frac{k_0+k_0'}{k_0-k_0'}\right|, \tag{20}$$

where it may be quite ambiguous even for the dropped integrals of finite value to remain finite in the limit of $\lambda_0=0$. The integral in (20) is easily seen to become of divergence according to the integration path in the two dimensional domain of k and k'. Thus we may conclude that the fourth order self energy of a free electron may be considered to be hardly possible to be made finite in contradistinction to Pauli's rather qualitative consideration.

§3. Lamb shift

For the accurate treatment of the Lamb shift it is required to carry out the relativistic treatment of a bound electron as shown in Tomonaga and Schwinger's covariant formalism. In order to investigate whether Dirac's cutting-off involved in his new method of field-quantization may actually give rise to the Lamb shift or not, we have here worked out, as a preliminary, the non-relativistic theory of the Lamb shift by the use of Dirac's new procedure instead of Bethe's artificial cutting-off.

As is well-known, the electromagnetic self-energy to the second order of an electron in the Coulomb field may be given by

$$\Delta W_m^r = \frac{e^2\hbar}{4\pi^2 cm^2}\int dk \sum_{n,s_i} \frac{|V_{mn}(k)|^2}{k_0(E_n - E_m + c\hbar k_0)}, \tag{21}$$

where n denotes each of the intermediate states of an electron, the summation over n stands for the discrete or continuous values of n according to whether there appear discrete or continuous states of the hydrogen atom, $s_i (i=1, 2)$ represents the polarization unit vector corresponding to each of the transverse waves, E_m and E_n are the energies of the stationary states m and n. Writing

the corresponding eigenfunctions by $\psi_m(x)$ and $\psi_n(x)$, we get

$$V_{mn}(\boldsymbol{k})=\int\psi_m^*(x)(\boldsymbol{s}_i\boldsymbol{p})e^{i\boldsymbol{k}\boldsymbol{x}}\psi_n(x)dx. \tag{22}$$

Introducing into (21) the λ-process and the negative energy photon according to Dirac's new procedure, it follows

$$\varDelta W_m=\frac{e^2\hbar}{4\pi^2cm^2}\int d\boldsymbol{k}\sum_{n,s_i}\frac{1}{2}\left\{\frac{|V_{mn}(\boldsymbol{k})|^2}{k_0(E_n-E_m+c\hbar k_0)}-\frac{|V_{mn}(-\boldsymbol{k})|^2}{k_0(E_n-E_m-c\hbar k_0)}\right\}\cos(\boldsymbol{\lambda k}-\lambda_0k_0). \tag{23}$$

Using the relation

$$|V_{mn}(\boldsymbol{k})|^2=\iint(\boldsymbol{s}_i\boldsymbol{p}^*\psi_m(x))^*\psi_n(x)\psi_n^*(x')e^{i\boldsymbol{k}(x-x')}(\boldsymbol{s}_i\boldsymbol{p}'^*\psi_m(x'))dxdx',$$

the equation (23) may be transformed into the following form

$$\varDelta W_m=\frac{e^2\hbar}{4\pi^2cm^2}\int_0^\pi\sin\theta d\theta\int_0^{2\pi}d\Phi\cdot\frac{1}{2}\cdot\sum_{n,s_i}\iint dxdx'(\boldsymbol{s}_i\boldsymbol{p}^*\psi_m(x))^*\psi_n(x)\psi_n^*(x')\cdot$$
$$\cdot(\boldsymbol{s}_i\boldsymbol{p}'^*\psi_m(x'))\left[\int_0^\infty\left\{\frac{k_0e^{i\boldsymbol{k}(x-x')}}{E_n-E_m+c\hbar k_0}-\frac{k_0e^{-i\boldsymbol{k}(x-x')}}{E_n-E_m-c\hbar k_0}\right\}\cos(\lambda_0k_0)dk_0\right], \tag{24}$$

where the integration with respect to photon momenta is expressed by polar coordinates ($d\boldsymbol{k}=k_0^2dk_0\sin\theta d\theta d\Phi$) and $\boldsymbol{\lambda}=0$ which means the adoption of a specified Lorentz frame. The last integral with respect to k_0 may easily be carried out by the use of the contour integral method and takes the following form

$$-\frac{i\pi}{2}(E_n-E_m)\left\{e^{-i\left\{\frac{\boldsymbol{k}}{k_0}(x-x')+\lambda_0\right\}(E_n-E_m)}-e^{-i\left\{\frac{\boldsymbol{k}}{k_0}(x-x')-\lambda_0\right\}(E_n-E_m)}\right\}, \tag{25}$$

which is valid under the condition of $-\lambda_0<\frac{\boldsymbol{k}}{k_0}(x-x')<\lambda_0$. The mentioned condition does not destroy the generality of our result for the reason that the most predominant contribution of the integral with respect to x' comes from the region of $x'=x$.

Putting (25) and $E_m\psi_m=H\psi_m$ into (24), and further, carrying out the x'-integration, we get, after performing the sum over n in accordance with the normalization condition $\sum_n\psi_n(x)\psi_n^*(x')=\delta(x-x')$,

$$\varDelta W_m=-\frac{\hbar e^2}{4\pi^2cm^2}\int_0^\pi\sin\theta d\theta\int_0^{2\pi}d\Phi\frac{\pi}{2}\sum_{s_i}\int dx\psi_m^*(x)\boldsymbol{s}_i\boldsymbol{p}\sin\{\lambda_0(H-E_m)\}(H-E_m)\boldsymbol{s}_i\boldsymbol{p}^*\psi_m(x), \tag{26}$$

which is reduced to the following form

$$\varDelta W_m=-\frac{\hbar e^2}{4\pi^2cm^2}\int_0^\pi\sin\theta d\theta\int_0^{2\pi}d\Phi\frac{\pi}{2}\sum_{s_i,n}a_n^*a_n\sin\{\lambda_0(E_n-E_m)\}(E_n-E_m), \tag{27}$$

by making use of

$$s_i p\psi_m(x) = \sum_n a_n \psi_n(x), \quad a_n = \int \psi_n^*(x) s_i p\psi_m(x) dx$$
$$= i\hbar \frac{m}{\hbar^2}(E_m - E_n) s_i \int \psi_n^*(x) x \psi_m(x) dx, \tag{28}$$

and the orthogonal relation

$$\int \psi_n^*(x)\psi_{n'}(x) dx = \delta_{nn'}. \tag{29}$$

The Lamb-shift due to the reaction of the electromagnetic interaction with the vacuum fluctuation may be considered to be given by (27), the electromagnetic mass term of the second order being equal to zero according to Pauli and Dirac.

To proceed further our computation, we shall take, in the following work, the Lamb-shift of $2S$-level of hydrogen atom, in which case we have

$$\sum_{s_i} a_n^* a_n = \left(\frac{m}{\hbar}\right)^2 (E_n - E_m)^2 \left[\frac{2}{3}(1 - \sin^2\theta \sin^2\Phi) + \frac{1}{3}(1 - \cos^2\theta)\right] (r_{2,0}^{n,1})^2, \tag{30}$$

and

$$r_{2,0}^{n,1} = \int_0^\infty R_{n,1}(r) r R_{2,0}(r) r^2 dr, \tag{31}$$

$R_{n,1}(r)$ being the radial eigenfunction of the hydrogen atom.

Putting (30) into (27), we get finally

$$\Delta W_{2,0} = \frac{-\hbar e^2}{4\pi^2 c m^2} \frac{4}{3}\pi^2 \left(\frac{m}{\hbar}\right)^2 \sum_n (r_{1,0}^{n,1})^2 (E_n - E_2)^3 \sin\{\lambda_0(E_n - E_2)\}. \tag{32}$$

For the case of discrete states with negative energies the matrix elements of (31) are easily evaluated with the hydrogen eigenfunction, which results are of the following form[9].

$$(r_{2,0}^{n,1})^2 = C \cdot \frac{2^{17} n^7 (n^2-1)(n-2)^{2n-6}}{(n+2)^{2n+6}}. \tag{33}$$

Substituting (33) into (32), we get

$$\Delta W_{2s,\,\mathrm{disc.}} = \frac{-\hbar e^2}{4\pi^2 c m^2} \frac{4\pi^2}{3}\left(\frac{m}{\hbar}\right)^2 \sum_n C \cdot \frac{n^7(n^2-1)(n-2)^{2n-6}}{(n+2)^{2n+6}} (E_n - E_2)^3 \sin\{\lambda_0(E_n - E_2)\}$$
$$= \quad '' \quad \sum_n \quad '' \quad \left(-\frac{1}{2}\left(\frac{Z}{a}\right)^2 \frac{1}{n^2} + \frac{1}{2}\left(\frac{Z}{a}\right)^2 \frac{1}{2^2}\right)^3 \cdot$$
$$\cdot \sin\left\{\frac{\lambda_0}{2}\left(\frac{Z}{a}\right)^2 \left(-\frac{1}{n^2} + \frac{1}{2^2}\right)\right\}, \tag{34}$$

in which, for finite n however large, the summands in (34) approach zero term by term in the limit of $\lambda_0 = 0$, while, for the range of infinitely large n, the

series in (34) may be approximately reduced to

$$\left(\frac{1}{2}\left(\frac{Z}{a}\right)^2\frac{1}{2^2}\right)^3\sin\left\{\frac{\lambda_0}{2}\left(\frac{Z}{a}\right)^2\frac{1}{2^2}\right\}\sum_n^\infty\frac{1}{n^3},$$

which is also clearly seen to vanish in the limit of $\lambda_0=0$.

Thus we have

$$\Delta W_{2s,\mathrm{disc.}}=0. \tag{35}$$

The summation over the continuous values of n is, for convenience, divided into two parts; the one corresponding to the n values of large positive energies while the other to those of small positive energies respectively. For the former the matrix elements (31) may be evaluated approximately with the asymptotic eigenfunction[9]

$$R_{E,1}=\frac{C}{kr}\sin\left(kr-\frac{\pi}{2}\right), \tag{36}$$

where k denotes the electron momentum related with the energy by $E=\hbar^2k^2/2m$, and for the latter the accurate eigenfunction[9] of

$$R_{E,1}=\frac{\sqrt{2Z/a}}{\sqrt{1-e^{-2\pi n'}}}\sqrt{1+n'^2}\,\frac{(2kr)}{3\,!}e^{ikr}F(in'+2,\,4,\,2ikr) \tag{36'}$$

may be taken, where $n'=(ak)^{-1}$, $a=$Bohr's radius and $F(\alpha,\beta,x)$ represents the confluent hypergeometric function. Then the summation over n may be replaced by the integration with respect to E, and then we have

$$\frac{-\hbar e^2}{4\pi^2cm^2}\left\{\frac{1}{2}\frac{\sin\lambda_0E_0}{E_0^{\frac{1}{2}}}+\frac{1}{2}\lambda_0^{\frac{1}{2}}\left[\sqrt{\frac{\pi}{2}}-\int_0^{\lambda_0E_0}\frac{\cos x}{x^{\frac{1}{2}}}dx\right]\right\}, \tag{37}$$

for the former case of large positive energies, in which E_0 denotes the minimum energy for the expression (36) to be valid approximately, and, further, for the latter case of small positive energies ($<E_0$), it follows

$$\frac{-\hbar e^2}{4\pi^2cm^2}\frac{4\pi^2}{3}\left(\frac{m}{\hbar}\right)^2\int_0^{E_0}dE\left|\frac{(Z/a)^2}{\sqrt{1-e^{-2\pi n'}}}\sqrt{1+n'^2}\,\frac{k}{3}\left[\frac{4\,!}{\left(ik-\frac{Z}{2a}\right)^5}\left(-\frac{ik+Z/2a}{ik-Z/2a}\right)^{-in'-}\left(1-\right.\right.\right.$$

$$\left.-\frac{2-in'}{4}\frac{2ik}{ik-\frac{Z}{2a}}\right)-\frac{Z}{2a}\frac{5\,!}{\left(ik-\frac{Z}{2a}\right)^6}\left(-\frac{ik+Z/2a}{ik-Z/2a}\right)^{-in'-1}\left(1-\frac{(2-in')2}{4}\frac{2ik}{ik-\frac{Z}{2a}}+\right.$$

$$\left.\left.\left.+\frac{(2-in')(3-in')2.1}{4.5\cdot 2\,!}\left(\frac{2ik}{ik-Z/2a}\right)^2\right)\right]\right|^2(E_n-E_2)^3\sin\{\lambda_0(E_n-E_2)\}. \tag{38}$$

Both expressions (37) and (38) may be readily shown to become zero in the limit of $\lambda_0=0$, which leads to

$$\Delta W_{2s,\mathrm{contl.}}=0. \tag{39}$$

Thus, we obtain by an application of Dirac's new procedure of field quantization to the second order self energy of a bound electron,

$$\Delta W_{2s}=0, \tag{40}$$

being in a drastic disagreement with the experimental result of Lamb and Retherford.

In conclusion it may be inferred from our results obtained above that Dirac's new procedure of field-quantization seems to cancell out all effects of the second order self-energies but fail completely to make finite the effects of the higher order self-energies in the one electron treatment.

The work here reported has been supported by the Scientific Research Expenditure of the Education Ministry.

References

1) H. A. Bethe, Phys. Rev. **72** (1947), 339; J. Schwinger, Phys. Rev. **74** (1948), 1439; **75** (1949), 651; **76** (1949), 790; S. Tomonaga and his coworkers, Prog. Theor. Phys. **3** (1948), 391; **4** (1949), 47, 121.
2) W. E. Lamb and R. C. Retherford, Phys. Rev. **72** (1947), 241; **75** (1949), 1325; **78** (1950), 539.
3) J. E. Nafe, E. B. Nelson and I. I. Rabi, Phys. Rev. **71** (1947), 914; J. E. Nafe and P. Kusch, Phys. Rev. **73** (1948), 584; J. E. Nafe and E. B. Nelson, Phys. Rev. **73** (1948), 718; R. Kusch and H. M. Foley, Phys. Rev. **74** (1948), 250.
4) W. H. Furry and M. Neuman, Phys. Rev. **76** (1949), 1677.
Independently of the computations of Neuman and Furry we have worked out the vacuum polarization in the vector meson theory in order to investigate the limit of validity of Tomonaga and Schwinger's renormalization procedure, the results of which computation have already been read in the symposiums on the theory of elementary particles held at Kyoto (1949) and Tokyo (1950) but not yet published.
5) W. Pauli, Rev. Mod. Phys. **15** (1943), 175; P. A. M. Dirac, Proc. Roy. Soc. **A180** (1942), 1.
6) W. Pauli, loc. cit.
7) H. A. Bethe, loc. cit.; N. M. Kroll and W. E. Lamb, Phys. Rev. **75** (1949), 388; J. B. French and V. F. Weisskopf, Phys. Rev. **75** (1949), 1240; H. A. Bethe, L. M. Brown and J. R. Stehn, Phys. Rev. **77** (1950), 370; S. Tomonaga and his coworkers, loc. cit.
8) H. A. Bethe, loc. cit.
9) H. A. Bethe, *Handbuch der Physik* **24**/I (1933), 274.

Letters to the Editor

Note on the Infrared Catastrophe

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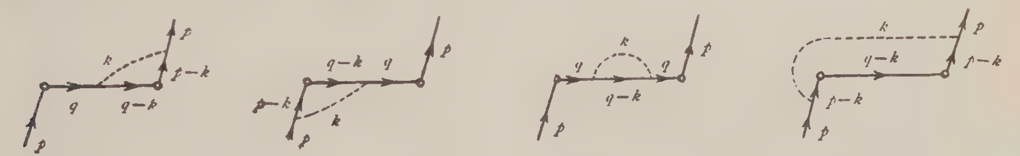
October 29, 1950

It is well-known that the so-called "infrared catastrophe" which one encounters in the evaluation of various processes involving real as well as virtual quanta is, as a matter of fact, an apparent difficulty resulting from inappropriate treatment of low frequency quanta and it always disappears when one takes into accout all relating processes that give contributions to the cross section considered[1]. But there has been no convincing explanation why such cancellation of infrared catastrophe always occurs. In this note we want to give some results of considerations about the mechanism of this phenomenon from the view point of the Tomonaga-Schwinger theory.

As the simplest example, we shall first be concerned with the second order radiative correction to the elastic scattering of an electron scattered by a fixed center of force. This process consists of two parts: 1) An incident electron (with momentum p_μ, $\mu=1, 2, 3, 4$) is scattered by the scattering center into a final state $(q_\mu-k_\mu)$ accompanied by an emission of a light quantum (k_μ). Squaring the marix element for this process one obtains a cross section of the order $e^2/\hbar c$. 2) During the incident electron (p_μ) is scattered into a final one (q_μ), a virtual quantum is emitted and reabsorbed by this electron and it does not exist in the final state. This process gives an $e^2/\hbar c$ modification of the matrix element for the elastic scattering and is combined with the latter to give an $e^2/\hbar c$ correction to the original cross section. As is easily seen from the unitarity of the S-matrix, the sum of these cross sections is proportional to the collection of real parts of the matrix elements corresponding to the four diagrams in Fig. 1. Each diagram involves both processes described above, and, therefore, it is expected that the divergences of matrix elements at the low frequency end of the light quantum cancel themselves ineach case. To see that this is really so, let us fix our eyes on the matrix element corresponding to the diagram in the Fig. 1a. This matrix element is, according to the Feynman's rule[2], proportional to



a b Fig. 1. c d

$$(-i)^{\frac{n}{2}}\int dk\ \bar{u}(p)\gamma_\mu\frac{i\gamma(p-k)-\kappa}{(p-k)^2+\kappa^2-i\varepsilon}\gamma_\nu A_\nu^e(p-q)\frac{i\gamma(q-k)-\kappa}{(q-k)^2+\kappa^2-i\varepsilon}\gamma_\mu\frac{i\gamma q-\kappa}{q^2+\kappa^2-i\varepsilon}\gamma_\lambda \times A_\lambda^e(q-p)u(p)\frac{1}{k^2-i\varepsilon}. \tag{1}$$

(n is the number of vertices which is 4 in this case; A_ν^e is a Fourier component of the scattering potential.) Evidently the non-vanishing term of the real part of (1) is

$$\pi^2\int dk\ \bar{u}(p)\gamma_\mu(i\gamma(p-k)-\kappa)\gamma_\nu A_\nu^e(p-q)$$
$$\times(i\gamma(q-k)-\kappa)\gamma_\mu(i\gamma q-\kappa)\gamma_\nu A_\nu^e(q-p)u(p)$$

$$\times\Big[((p-k)^2+\varkappa^2)^{-1}\delta((q-k)^2+\varkappa^2)(q^2+\varkappa^2)^{-1}$$
$$\times\delta(k^2)$$
$$+((p-k)^2+\varkappa^2)^{-1}((q-k)^2+\varkappa^2)^{-1}\delta(q^2+\varkappa^2)$$
$$\times\delta(k^2)$$
$$+((p-k)^2+\varkappa^2)^{-1}\delta((q^2-k^2)+\varkappa^2)\delta(q^2+\varkappa^2)$$
$$\times(k^2)^{-1}$$
$$+\delta((p-k)^2+\varkappa^2)((q-k)^2+\varkappa^2)^{-1}\delta(q^2+\varkappa^2)$$
$$(k^2)^{-1}\Big]$$

since

$$\lim_{\varepsilon\to+0}\frac{1}{x-i\varepsilon}=P\left(\frac{1}{x}\right)+i\pi\ \delta(x). \qquad (2)$$

(P means the principal value.) The first term in the square bracket gives the cross section for emission of a real photon and the other three terms give contributions coming from virtual one. We shall now limit ourselves to the domain of integration in which $k_\mu\sim 0$. As is easily seen from (2), the third and the fourth terms which do not involve a factor $\delta(k^2)$ are convergent at the low frequency end and can be omitted from the following considerations. In the numerator of (2), moreover, we can safely neglect terms which involve several k_μ's. Thus (2) can be simplified to

$$\pi^2\bar{u}(p)\gamma_\mu(i\gamma p-\varkappa)\gamma_\nu A_\nu^e(i\gamma q-\varkappa)\gamma_\mu(i\gamma q-\varkappa)$$
$$\gamma_\lambda A_\lambda^e\ u(p)$$

$$\times\int dk((p-k)^2+\varkappa^2)^{-1}\delta(k^2)\Big[(q^2+\varkappa^2)^{-1}\ \delta((q$$
$$-k)^2+\varkappa^2)+((q-k)^2+\varkappa^2)^{-1}\ \delta(q^2+\varkappa^2)\Big] \quad (4)$$

fo $k_\mu\sim 0$. This integral is now easily seen to be convergent for $|\boldsymbol{k}|-0$:

$$\int dk((p-k)^2+\varkappa^2)^{-1}\delta(k^2)\Big[(q^2+\varkappa^2)^{-1}\delta((q-$$
$$k)^2+\varkappa^2)+((q-k)^2+\varkappa^2)^{-1}\delta(q^2+\varkappa^2)\Big]$$

$$=\int dk(-2pk)^{-1}\delta(k^2)\Big[(2qk)^{-1}\delta(q^2+\varkappa^2-2qk)$$
$$+(2qk)^{-1}\delta(q^2+\varkappa^2)\Big]$$

$$\sim\int dk(-2pk)^{-1}\delta(k^2)(2qk)^{-1}(-2qk)\delta'(q^2+\varkappa^2)$$

$$\propto\int\frac{dk}{|\boldsymbol{k}|^2} \qquad (5)$$

where $p^2+\varkappa^2=0$ is employed. It is thus shown that there appears no infrared catastrophe in this process. The same result holds as well for other diagrams in the Fig. 1. (However, Fig. 1c has to be treated more carefully since it involves a graph of mass renormalization type.) It is easy to extend this method to the general cases involving arbitrary number of light quanta and to arrive at the same conclusion (irrespective of the type of particles interacting with the electromagnetic field).

The present analysis has made clear the reason why the infrared catastrophe once appears in every radiative process, and yet it always disappears when one takes all relating processes into account. The answer is: ***Because it does not exist at all from beginning.*** We have to remind ourselves, however, of the fact that another important feature of the infrared problem, that is, the vanishing of the cross section for emission of finite number of photons, is not touched at all in this treatment.

It is to be noted that the Feynman's method of integration[3] which is now extensively employed is not legitimate in a strict sense. This method consists in uniting the denominators of (1) into a single term making use of the obvious formula

$$a^{-1}b^{-1}=\int_0^1 du\ (au+b(1-u))^{-2}. \quad (6)$$

Then the origin of the k-space is translated so that one can utilize the spherical symmetry with respect to the integration variables. Such a translation, however, reduces the possible "displaced poles"[4] in the integrand of (1) to the ordinary poles and thus the

result of this integration may be different from the original (1) by a finite contribution arising from the residue of these poles. One can see with ease that the first term in the square bracket of (2) appears just in this way and would try to evaluate the Feynman's method without care.

Detailed account about this problem will be published in a later issue of this journal.

1) F. Bloch and A. Nordsieck, Phys. Rev. **52** (1937), 54.
   Z. Koba, Prog. Theor. Phys. **3** (1948), 208.
2) F. J. Dyson, Phys. Rev. **75** (1949), 1736.
3) R. P. Feynman, Phys. Rev. **76** (1949), 769.
4) F. J. Dyson, Phys. Rev. **65** (1949), 1736, esp. Section V.

Note on the Magnetic Resonance Absorption in Paramagnetic Salts

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November 10, 1950.

The paramagnetic anisotropy of copper sulfate $CuSO_4 \cdot 5H_2O$ was investigated by K.S. Krishnan and A. Mookherji,[1] and its crystal structure was determined by C. A. Beevers and H. Lipson.[2] From their works it was found that this substance contains two copper ions in a unit which are each situated in the center of the octahedron constructed from four H_2O molecules at the corners of a square and two oxygen ions on a straight line drawn perpendicular to it through its center. Therefore, each copper ion is situated at a center of tetragonal symmetry. The tetragonal axes of two octahedrons in a unit cell are not parallel but are almost perpendicular to each other so that all the copper ions in the crystal are divided into two groups. As each group has the tetragonal symmetry with respect to its own tetragonal axis, the magnetic anisotropy of the whole crystal be derived from that of the system of the two tetragonal groups whose orientations are mutually different with respect to each other. Based on this model D. Polder[3] calculated the paramagnetic susceptibilities of this substance, the values of which are in good accordance with experimental ones.

Now let Z_2 and Z_5 be the two tetragonal axes and 2ϕ the angle between them. The two bisectors of the angle between Z_1 and Z_2 in the plane containing them and the axis perpendicular to that plane are then three magnetic principal axes α, β, γ. The three principal Landé factors g_α, g_β, g_γ are given by

$$g_\alpha^2=g_\parallel^2\cos^2\phi+g_\perp^2\sin^2\phi,$$
$$g_\beta^2=g_\parallel^2\sin^2\phi+g_\perp^2\cos^2\phi, \qquad (1)$$
$$g_\gamma=g_\perp,$$

where $g_\parallel$ and $g_\perp$ are respectively the Landé factors parallel and perpendicular to the tetragonal axis of each group. In general, using the angle θ_i between Z_i and the applied magnetic field $\boldsymbol{H}$, we obtain for the Landé factor parallel to $\boldsymbol{H}$

$$g_i^2=g_\parallel^2\cos^2\phi_i+g_\perp^2\sin^2\theta_i. \qquad (2)$$

This formula shows that in the case the magnetic field is applied perpendicular to β-axis, the g-values of the two tetragonal groups are equal but that they are different in the case $\boldsymbol{H}$ is perpendicular to γ-axis. Accordingly, two magnetic resonance lines should appear in the latter case, corresponding to two different values of g. These circumstances remain the same for $CuK_2(SO_4)_2 \cdot 6H_2O$ which has also two tetragonal groups.

In a microwave experiment with a wavelength of 3 cm, R. D. Arnold and A. F. Kip[4] found in fact that two resonance lines appear for $CuK_2(SO_4)_2 \cdot 6H_20$, although it is not the case for $CuSO_4 \cdot 5H_2O$. They pointed out that the absence of the separation of the resonance line for the latter case is due to the exchange coupling, since the resonance line width is more narrow and the nearest Cu-Cu distance smaller for $CuSO_4 \cdot 5H_2O$ than

for $CuK_2(SO_4)_2 \cdot 6H_2O$.

Recently, however, it was found with a shorter wave-length of 1cm that even in $CuSO_4 \cdot 5H_2O$ the resonance line separates. In similar substances like $CuK_2Cl_4 \cdot 2H_2O$ and $Cu(NH_4)_2\,Cl_4 \cdot 2H_2O$ the same effect is reported[5], namely that the resonance line separates for a sufficiently short wave-length. These results appear to show that it is the ratio of the strength of the exchange coupling to that of the resonance magnetic field, which is inversely proportional to the wave-length, which determines the separation of the resonance line.

To explain the above mentioned effect in a qualitative way, let us consider as a simplest model a system that consists of only two copper ions, belonging each to the two different tetragonal groups. The g-values are in general different for these two ions and are dependent on the angle between Z_t and the applied magnetic field direction.

The Hamiltonian of this two spin system with a magnetic field applied parallel to z-axis can be approximated as follows:

$$H=\beta H(g_1 S_{1z}+g_2 S_{2z})+2J(S_1 \cdot S_2), \quad (3)$$

where β, H, S, J denote, as usual, Bohr magneton, field strength, spin vector and exchange integral, and g_1 and g_2 are Landé factors for the two ions which are generally different. In this equation the first term represents the Zeeman energy and the second the exchange energy. Excluding constant terms, Eq. (3) can be written as

$$H=\beta g_1 H(S_z-\gamma S_{2z})+JS^2, \quad (3')$$

where

$$\gamma=1-\frac{g_2}{g_1}, \quad S=S_1+S_2.$$

In the case of $S_1=S_2=1/2$, as is the case for Cu^{++}, the eigenvalues of this Hamiltonian can immediately be obtained as follows:

$$S=0,\ S_z=0,\ E_{0,0}=J-(J^2+\frac{1}{4}\beta^2 g_1^2 H^2\gamma^2)^{1/2}$$

$$S=1,\ S_z=-1,\ E_{1,-1}=2J-\beta g_1 H(1-\frac{1}{2}\gamma).$$

$$S=1,\ S_z=0,\ E_{1,0}=J+\frac{1}{4}\beta^2 g_1^2 H^2\gamma^2)^{1/2}, \quad (4)$$

$$S=1,\ S_z=1,\ E_{1,1}=2J+\beta g_1 H(1-\frac{1}{2}\gamma).$$

On account of the selection rule for $S_{1z}+S_{2z}=S_z$, only two following transitions are allowed between the above four energy levels.

$(S=1,\ S_z=-1)\longrightarrow(S=1,\ S_z=0)$
A-transition,

$(S=1,\ S_z=0)\longrightarrow(S=1,\ S_z=1)$
B-transition.

From Eqs. (4), the energies absorbed by A- and B-transition, $h\nu_A$ and $h\nu_B$, are respectively given by

$$h\nu_A=E_{1,0}-E_{1,-1}=J+(J^2+\frac{1}{4}\beta^2 g_1^2 H^2\gamma^2)^{1/2}$$
$$-2J+\beta g_1 H(1-\frac{1}{2}\gamma),$$

$$h\nu_B=E_{1,1}-E_{1,0}=2J+\beta g_1 H(1-\frac{1}{2}\gamma)-J-$$
$$(J^2+\frac{1}{4}\beta^2 g_1^2 H^2\gamma^2)^{1/2}. \quad (5)$$

Thus, from these equations the following conclusions can be derived.

(i) $g_1=g_2$ i.e., $\gamma=0$ In this case we obtain

$$h\nu_A=h\nu_B=\beta g_1 H. \quad (6)$$

(ii) $J \ll (1/2)\beta g_1 H\gamma$ In this case, expanding the square root in power series of $2J/\beta\, g_1 H\gamma$, the following results are obtained.

$$\begin{aligned} h\nu_A&=\beta g_1 H-J+J^2/\beta g_1 H\gamma, \\ h\nu_B&=\beta g_2 H+J-J^2/\beta g_1 H\gamma. \end{aligned} \quad (7)$$

In the limiting case of J—0, we can expect two resonance lines $\beta g_1 H$ and $\beta g_2 H$ as far as g_1 is unequal to g_2.

(iii) $J \gg (1/2)\beta g_1 H\gamma$ In this case, expanding the square root in power series of $\beta g_1 H\gamma/2J$, we get the following results:

$$h\nu_A=\frac{1}{2}(g_1+g_2)\beta H+\frac{1}{2}J(\beta g_1 H\gamma/2J)^2,$$

$$h\nu_B=\frac{1}{2}(g_1+g_2)\beta H-\frac{1}{2}J(\beta g_1 H\gamma/2J)^2. \tag{8}$$

In the limiting case of $J=\infty$, we have only one resonance frequency $1/2(g_1+g_2)\beta H$, the arithmetic mean of the two resonance frequencies for the case (ii).

The transition from the case (ii) to the case (iii) is determined by the condition

$$J=\frac{1}{2}\beta g_1 H\gamma. \tag{9}$$

For $CuSO_4\cdot 5H_2O$, this transition point may exist between the wave-lengths 3cm and 1cm Assuming that this transition point exists at the wavelength of 2cm and substituting 0.33 for $g_1\gamma=g_1-g_2\sim g_{\parallel}-g_{\perp}$, we can estimate the exchange integral. The value of J obtained in this manner is $J=8.2\times10^{-18}$ erg, which agrees in its order of magnitude with the value estimated by A. Wright[7] from the resonance frequency distribution curve of $CuSO_4\cdot 5H_2O$.

In conclusion the writer wishes to express his cordial thanks to Mr. M. Fujimoto for his valuable discussions on this problem.

1) K. S. Krishnan and A. Mookherji, Phys. Rev. **50** (1936), 860; **54** (1938), 533; **54** (1938), 841.
2) C. A. Beevers and H. Lipson, Proc. Roy. Soc. London **146** (1934), 570.
3) D. Polder, Physica **9** (1942), 702.
4) R. D. Arnold and A. F. Kip, Phys. Rev. **75** (1949), 1199.
5) H. Kumagai and his coworkers, unpublished. M. Fujimoto and J. Ito, unpublished.
6) P. R. Weiss, Phys. Rev. **74** (1948), 1493.
7) A. Wright, Phys. Rev. **76** (1949), 1826.

Neutron Cross Section and Nuclear Shell Structure

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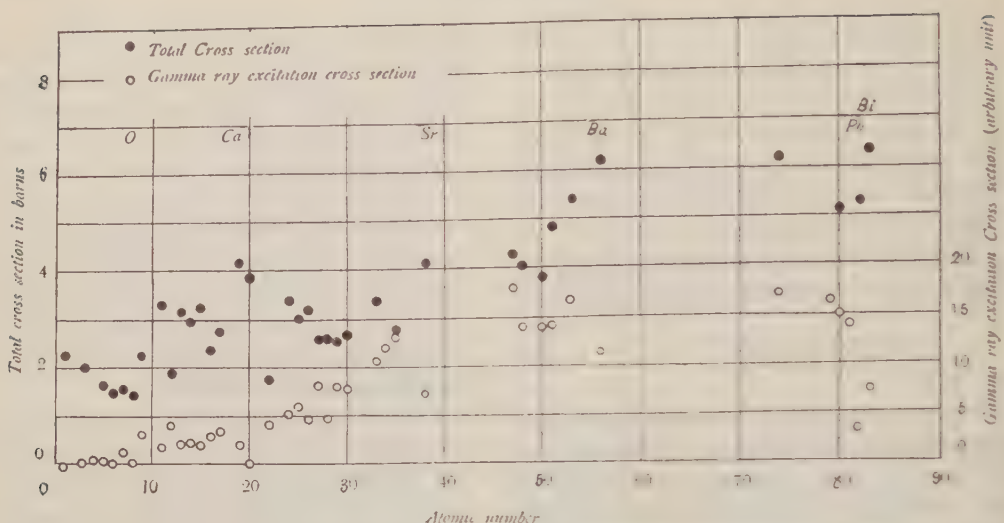
November 23, 1950

D. J. Hughes and D. Shermann[1] investigated the radiative capture process of various nuclei using the fission neutrons of effective energy 1 MeV. They expected that the neutron closed shell nucleus has small capture cross section of neutron because of its low binding energy for an additional neutron, low excitation energy after neutron capture, and hence small level density. Their experimental results indicate that the neutron closed shell nuclei containing 50, 82 or 126 neutrons have very small capture cross section of neutrons as they expected.

From similar point of view we will discuss the inelastic scattering of neutrons which is the other dynamical process of neutrons involving the formation of the compound nucleus. H. Aoki[2] measured the gamma ray excitaion cross sections of 2.5MeV *d-d* neutrons for 41 elements. These cross sections, which are shown in Fig. 1 with open circles, may be regarded to be proportional to the inelastic scattering cross sections, considering the number of gamma ray emitted and the efficiency of the detector. O, Ca, Sr, Ba, Pb and Bi, which are abundant in the isotope of neutron closed shell, have small gamma ray excitation cross sections. Thus we may say that the neutron closed shell nucleus has smaller inelastic scattering cross section than those of neighbouring nuclei. This may be accounted for by the situation that the neutron closed shell nucleus has small probability to form the compound nucleus with neutron because of wide level spacing.

S. Kikuchi and H. Aoki[3] also measured the total scattering cross sections of 37 elements using the same neutron source. These data are plotted in Fig. 1 with closed circles. It is interesting that Ca, Sr and Ba, neutron closed shell nuclei, have comparatively large total scattering cross sections, in spite of their small gamma ray excitation cross section.

In the above discussions we neglect the resonance effects by which the data may be affected. In fact, R. K. Adair et al[4] measured the total scattering cross section for Ca



using neutrons below 500 keV and obtained very small values in the wide energy range. They considered the results to be the effects of the neutron closed shell, together with the total scattering cross section for K in the same energy region[5].

However, using heterogeneous Li-d neutrons, the total scattering cross sections have been measured in our laboratory[6] and the results obtained are similar to those of d-d neutrons. The measurement of the gamma ray excitation cross sections of Li-d neutrons is now proceeded in our laboratory. More detailed discussion about the relation between the neutron cross section and the nuclear shell structure will be done in connection with this experiment.

1) D. J. Hughes and D. Shermann, Phys. Rev. **78** (1950), 632.
2) H. Aoki, Proc. Phys. Math. Soc. Japan **19** (1937), 370. See also I. Nonaka, ibid. **25** (1943), 227.
3) S. Kikuchi and H. Aoki, ibid. **21** (1939), 75.
4) R. K. Adair, H. H. Barschall, C. K. Bockelmann and O. Sala, Phys. Rev. **75** (1949), 508.
5) Fields, B. Russell, D. Sachs and A. Wattenberg, Phys. Rev. **71** (1947), 508.
6) T. Wakatsuki and K. Sugimoto, Jour. Phys. Soc. Japan (to be published).

Note on the Classical Equations of Motion of Nucleons

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December 1, 1950

According to the theory of action at a distance by Wheeler and Feynman,[1] the fields which act on a given particle arise only from other particles and these fields are represented by one half the retarded plus one half the advanced solutions of the field equations. We apply this theory to the nucleons interacting with meson fields.

The equations of motion are derived from the action principle that the action

$$J=\sum_a M_a\int (v_i^a v^{ai})^{1/2}d\tau_a+\sum_{a<b} g_a g_b \times\iint \bar{\Delta}(z^a-z^b)v_i^a v^{bi}d\tau_a d\tau_b \qquad (1)$$

is an extremum. Here τ_a, z_i^a and v_i^a denote respectively the proper time, the coordinates and the velocities of a nucleon of mass M_a and mesonic charge g_a. The symbol $x_i y^i$ means $x_0y_0-x_1y_1-x_2y_2-x_3y_3$. Further we put the light velocity $c=1$. $\bar{\Delta}(x-x')$ is a particular solution of the

following equation

$$(\Box+\varkappa^2)\bar{\Delta}(x-x')$$
$$=\delta(x_0-x_0')\delta(x_1-x_1')\delta(x_2-x_2')\delta(x_3-x_3). \qquad (2)$$

The explicit form of $\bar{\Delta}(x-x')$ is given by Schwinger[2)]

$$\bar{\Delta}(x-x')=\frac{-1}{(2\pi)^4}$$
$$\times P\iiiint\frac{e^{ik_i(x^i-x'^i)}}{k_ik^i-\varkappa^2}dk_0dk_1dk_2dk_3$$
$$=\frac{1}{4\pi}\delta(\lambda)-\frac{\varkappa^2}{8\pi}R_e\frac{H_1^{(1)}(\varkappa\lambda^{1/2})}{\varkappa\lambda^{1/2}} \qquad (3)$$

where $\lambda=(x_i-x_i')(x^i-x^{i\prime})$ and $\varkappa=\frac{m}{\hbar}$, m being the meson mass. Defining

$$\varphi_i^{(b)}(x)=g_b\int_{-\infty}^{+\infty}v_i^b\bar{\Delta}(x-z^b)d\tau_b, \qquad (4)$$

we can write (1) as

$$J=\sum_a M_a\int(v_i^a v^{ai})^{1/2}d\tau_a$$
$$+\sum_{a<b}g_a\int\varphi_i^{(b)}(z^a)v^{ai}d\tau_a.$$

Variating the path z_i^a of the particle a, we obtain the equations of motion:

$$M_a\dot{v}_i^a=g_a\sum_{b\neq a}\left(\frac{\partial\varphi_k^{(b)}(z^a)}{\partial z^{ai}}-\frac{\partial\varphi_i^{(b)}(z^a)}{\partial z^{ak}}\right)v^{ak}$$
$$=g_a\sum_{b\neq a}\varphi_{ik}^{(b)}(z^a)v^{ak}. \qquad (5)$$

From (2) and (4) we see that $\varphi_i^{(b)}(x)$ satisfies the equation

$$(\Box+\varkappa^2)\varphi_i^{(b)}(x)=4\pi j_i^{(b)}(x), \qquad (6)$$

where

$$j_i^{(b)}(x)=g_b\int_{-\infty}^{+\infty}v_i^b(\tau_b)\delta(x_0-z_0^b)\delta(x_1-z_1^b)$$
$$\times\delta(x_2-z_2^b)\delta(x_3-z_3^b)d\tau_b \qquad (7)$$

is the density-current four vecter due to particle b. The divergence of the vector potential $\varphi_i^{(b)}(x)$ vanishes:

$$\frac{\partial\varphi_i^{(b)}(x)}{\partial x_i}=g_b\int v_i^b\frac{\partial\bar{\Delta}(x-z^b)}{\partial x_i}d\tau_b$$
$$=g_b\int_{-\infty}^{+\infty}v_i^b\frac{\partial\lambda}{\partial x_i}\frac{d}{d\lambda}\bar{\Delta}(\lambda)\ d\tau_b$$
$$=g_b\int_{-\infty}^{+\infty}2(x^i-z^{bi})v_i^b\frac{d}{d\lambda}\bar{\Delta}(\lambda)\ d\tau_b$$
$$=-g_b\int_{-\infty}^{+\infty}\frac{d\lambda}{d\tau_b}\frac{d}{d\lambda}\bar{\Delta}(\lambda)\ d\tau_b$$
$$=-g_b\left[\frac{1}{4\pi}\delta(\lambda)-\frac{\varkappa^2}{8\pi}R_e\frac{H_1^{(1)}(\varkappa\lambda^{1/2})}{\varkappa\lambda^{1/2}}\right]_{-\infty}^{+\infty}=0. \qquad (8)$$

Accordingly $\varphi_{ik}^{(b)}(x)$ satisfies the field equation

$$\frac{\partial\varphi_{ik}^{(b)}(x)}{\partial x_k}-\varkappa^2\varphi_i^{(b)}(x)=-4\pi j_i^{(b)}(x). \qquad (9)$$

The fields (4) are half the sum of the retarded and advanced potentials of particle b:

$$\varphi_i^{(b)}(x)=\frac{1}{2}\left\{\frac{g_bv_i^b}{v_k^b(x^k-z^{bk})}\bigg|_{\tau_b^0}-\frac{g_b\varkappa^2}{2}\int_{-\infty}^{\tau_b^0}v_i^b\frac{J_1(\varkappa\lambda^{1/2})}{\varkappa\lambda^{1/2}}d\tau_b\right\}$$
$$+\frac{1}{2}\left\{\frac{g_bv_i^b}{v_k^b(x^k-z^{bk})}\bigg|_{\tau_b'^0}-\frac{g_b\varkappa^2}{2}\int_{\tau_b'^0}^{+\infty}v_i^b\frac{J_1(\varkappa\lambda^{1/2})}{\varkappa\lambda^{1/2}}d\tau_b\right\}$$
$$=\frac{1}{2}\{\varphi_i^{(b)}(x)_{\mathrm{ret}}+\varphi_i^{(b)}(x)_{\mathrm{adv}}\}, \qquad (10)$$

where τ_b^0 and $\tau_b'^0$ are the retarded and advanced proper times and $\lambda=(x_i-z_i^b)\times(x^i-z^{bi})$.

Now the equations of motion can be written as

$$M_a\dot{v}_i^a=g_av^{ak}\Big\{\frac{1}{2}(\varphi_{ik}^{(a)}(z^a)_{\mathrm{ret}}-\varphi_{ik}^{(a)}(z^a)_{\mathrm{adv}})$$
$$+\sum_{b\neq a}\varphi_{ik}^{(b)}(z^a)_{\mathrm{ret}}+\frac{1}{2}\sum_b(\varphi_{ik}^{(b)}(z^a)_{\mathrm{adv}}$$
$$-\varphi_{ik}^{(b)}(z^a)_{\mathrm{ret}})\Big\} \qquad (11)$$

The case which Wheeler and Feynman call "complete absorption" is characterized by

$$\sum_b(\varphi_{ik}^{(b)}(x)_{\mathrm{ret}}-\varphi_{ik}^{(b)}(x)_{\mathrm{adv}})=0. \qquad (12)$$

In this case (11) becomes

$$M_a\dot{v}=\frac{g_av^k}{2}(\varphi_{ik}^{(a)}(z^a)_{\mathrm{et}}-\varphi_{ik}^{(a)}(z^a)_{\mathrm{adv}})$$

$$+g_a v^k \sum_{b\neq a} \varphi_{ik}^{(b)}(z^a)_{\rm ret}$$

$$=\frac{2}{3}g_a^2(\ddot{v}_i+\dot{v}^2 v_i)+\frac{g_a^2\varkappa^2}{2}v^k$$

$$\times\left\{\int_{-\infty}^{\tau_a}\frac{s_i v_k - s_k v_i}{s^2}J_2(\varkappa s)d\tau_a'\right.$$

$$\left.-\int_{\tau_a}^{+\infty}\frac{s_i v_k - s_k v_i}{s^2}J_2(\varkappa s)d\tau_a'\right\}$$

$$+g_a v_k \sum_{b\neq a}\varphi_{ik}^{(b)}(z^a)_{\rm ret}, \qquad (13)$$

where $s_i=z_i^a(\tau_a)-z_i^a(\tau_a')$ and $s^2=s_i s^i$.

Whereas the field theoretical calculation leads to the following equations of motion :[3]

$$M_a\dot{v}_i=\frac{2}{3}g_a^2(\ddot{v}_i+\dot{v}^2 v_i)+g_a^2\varkappa^2 v^k$$

$$\times\int_{-\infty}^{\tau_a}\frac{s_i v_k - s_k v_i}{s^2}J_2(\varkappa s)d\tau_a'$$

$$+g_a v^k\sum_{b\neq a}\varphi_{ik}^{(b)}(z^a)_{\rm ret}. \qquad (14)$$

The appearance of the terms in the curly bracket in (13) or the third term in (14) is related to the fact that the Huygens principle does not hold for the fields with non-vanishing rest masses. The terms in the curly bracket in (13) mean that the future as well as the past has influence upon the present.

Next we consider the scattering of mesons by nucleons. We treat the case that the external forces acting on the nucleon have the property of the harmonic oscillation along x_1-axis : $\varphi_{10}^{\rm ext}=\gamma\cos\omega_0 t$, $\varphi_{20}^{\rm ext}=\varphi_{30}^{\rm ext}=\cdots=0$ and that the velocity of the nucleon is small. The solutions of (14) in this case are given by Bhabha.[3] We write the solutions of (13) as

$$z_1=\frac{\beta}{w_0}\sin(\omega_0 t+\delta),\ z_2=z_3=0,\ z_0=t.$$

Here we assume that $\beta\ll 1$ and β^2 can be neglected. Then $v_1=\frac{dz_1}{dt}\left\{1-\left(\frac{dz_1}{dt}\right)^2\right\}^{-1}$ $\fallingdotseq\beta\cos(\omega_0 t+\delta)$, $v_0=\left\{1-\left(\frac{dz_1}{dt}\right)^2\right\}^{-1}\fallingdotseq 1$ and in the integrals in (13)

$$s_0=z_0(\tau)-z_0(\tau')\fallingdotseq t-t',\ s_1=z_1(\tau)-z_1(\tau')$$

$$=\frac{\beta}{\omega_0}\left\{\sin(\omega_0 t+\delta)-\sin(\omega_0 t'+\delta),\right\}$$

$$s=\sqrt{s_0^2-s_1^2}\fallingdotseq t-t'.$$

In this approximation (13) takes the form,

$$-a\omega_0\beta\sin(\omega_0 t+\delta)=-\omega_0^2\beta\cos(\omega_0 t+\delta)$$

$$+\omega_0^2\beta P\cos(\omega_0 t+\delta)+\frac{3\gamma}{2g}\cos(\omega_0 t), \qquad (15)$$

where $a=\frac{3M}{2g^2}$ and $P=\frac{3}{2}\frac{\varkappa^2}{\omega_0^2}\times$ $\left\{\frac{1}{\omega_0}\int_0^\infty\frac{J_2(\varkappa s)}{s^2}\sin\omega_0 s ds-\int_0^\infty\frac{J_2(\varkappa s)}{s}\cos\omega_0 s ds\right\}$.

β and $\cos\delta$ are determined from (15)

$$\beta=\frac{3\gamma}{2g\omega_0\{a^2+(1-P)^2\omega_0^2\}^{1/2}},$$

$$\cos\delta=\frac{(1-p)\omega_0}{\{a^2+(1-P)^2\omega_0^2\}^{1/2}}.$$

If $\omega_0<\varkappa$, no energy is radiated by the nucleon and the whole field moves in phase with nucleon. In this case the effective mass of the nucleon is $\frac{2}{3}g^2\{a^2+(1-P)^2\omega_0^2\}^{1/2}$ which reduces to M if $\omega_0\ll\varkappa$. On the other hand (14) gives in this limit as the effective mass $M-\frac{g^2}{2}\varkappa$.[3] The above result coincides with that there is no self-energy according to the theory of action at a distance.

1) J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. **17** (1945), 157; **21** (1949), 425.
2) J. Schwinger, Phys. Rev. **75** (1949), 651.
3) H. J. Bhabha, Proc. Roy. Soc. **172** (1939), 384.

Photo-Meson Production and Nucleon Isobar

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December 11, 1950

The experiments on the production of mesons by X-rays[1] have revealed (1) that the neutral mesons are produced with cross sections similar to those for the charged mesons, (2) that the angular distribution in centre of mass system is nearly uniform and (3) that the excitation

function flatterns off for photon energies above about 250 Mev. The production cross section of hydrogen has been calculated by many authors, but all theories have been unsuccessful in explaining these phenomena : the second order perturbation theory[2] predicts too small yield for π^0 as compared to π^+. If the contribution from the anomalous magnetic moment of nucleon is taken into account[3], the total cross section for π^0-production becomes comparable to that for π^+, but the angular and energy dependence does not agree with the experiments.

It is the purpose of this letter to point out that the situation is much improved if one assumes the existence of proton isobar. In the strong coupling meson theory, this appears as the excited state of higher spin or isotopic spin, and causes resonance in our case. Thus the proton, absorbing a photon, is excited to its isobar state, followed by the emission of charged or neutral meson. In the case of charge symmetry, the emission of charged and neutral meson from isobar will be equally probable, thus satisfying the first condition cited above. If the resonance level and total width of the excited state are suitably adjusted, the excitation function can be reproduced well.

The calculations are carried out for pseudoscalar symmetrical meson under the assumption of strong coupling*. For simplicity, we shall neglect the small oscillation near equilibrium position, i.e., those described by p and q in Pauli and Dancoff's paper[4]. The isobar separation E_0 turns out to be $a\varkappa^2 a/g^2$, where g means the pseudo-vector coupling constant, and $\varkappa$ the meson mass, and a the cut off radius, all in natural units. a is a constant depending on the shape of the source function, which may be approximately put equal to 5, as is done in the following. To fit the experimental data, one has to put $E_0 \sim 240$Mev, and the result is $\varkappa a/g^2 \sim 1/3$. With this value, the neutron magnetic moment becomes 1.4 nuclear magneton, which is a reasonable agreement with the observation. The total width of the excited state, due to the emission of mesons, is

$$\Gamma = \frac{27}{8}\left(\frac{\varkappa a}{g}\right)^2 \frac{p_0^3}{E_0^2}$$

where

$$p_0^2 = E_0^2 - \varkappa^2.$$

There is another contribution from γ-emission, which is, however, of negligible amount. The cross sections of proton for neutral and charged meson production are,

$$\sigma(\pi^0) = \frac{1}{12^2}\left(\frac{eg}{\varkappa}\right)^2 \frac{p^3}{4E} \frac{2+3\sin^2\theta}{(E-E_0)^2 + \frac{1}{4}\Gamma^2} d\Omega,$$

$$\sigma_1(\pi^+) = \frac{1}{2}\sigma(\pi^0),$$

respectively, where E is the energy of incident photon and p is the momentum of emitted meson, that is $p^2 = E^2 - \varkappa^2$. For charged meson, owing to its current, there is another term

$$\sigma_2(\pi^+) = \frac{1}{9}\left(\frac{eg}{\varkappa}\right)^2 \left(\frac{p}{E} - \frac{2\varkappa^2 p^3}{E\omega^4}\sin^2\theta\right) d\Omega,$$

$$\omega^2 = 2E^2 + 2pE\cos\theta$$

together with the cross terms of rather complicated nature. The latter, however, is not of importance.

Taking a about three times of the nucleon Compton wave length, we have $g^2 \sim 1$, and $\Gamma \sim 50$ Mev. The total cross section for at an incident energy of 240 Mev, is $\sim 1.5 \times 10^{-28}\text{cm}^2$. These values are in good agreement with the experiments.

In conclusion we should emphasize the fact that the $\gamma - \pi$ data are in favour of the possible existence of nucleon isobar with an excitation energy of about $1.7\varkappa$. This value is not very far from $2\varkappa$ obtained by Marshak from other considerations.

* The calculation of Watson and Hart (Phys. Rev. **79**(1950), 918) for intermediate as well as strong coupling completely neglects the effect of isobar, with the result exhibiting no resonance.

1) J. Steinberger, W.K.H. Panofsky and J. Steller, Phys. Rev. **78** (1950), 802; A. S. Bishop, J. Steinberger and L. J. Cook, Phys. Rev. **80** (1950), 291.
2) G. Araki, Prog. Theor. Phys. **5** (1950), 507 and other papers cited in this paper.
3) K. Aidzu, Y. Fujimoto and H. Fukuda, Prog. Theor. Phys. to be published.
4) W. Pauli and S. M. Dancoff, Phys. Rev. **62** (1942), 85.
5) R. E. Marshak, Phys. Rev. **78** (1950), 346.

Note on the Energy-Momentum Tensor

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December 12, 1950

Previously, we proposed one method[1] to remove the ambiguities which appear in the present quantum field theory, for instance the difficulties of gauge invariancy, divergence and equivalence theorem. To clarify the meaning of this method, we can take the following equation as simple example:

$$\lim_{x\to 0} \prod_{i=1}^{n} x_{\mu_i} S_F(x)=0, \quad (n=1, 2, 3, \cdots\cdots). \tag{1}$$

This equation does not hold owing to the singular function unless one regularize this function by any method and this fact is the very origin of the difficulties of ambiguity. For these circumstances, we are obliged to utilize the following conditions to avoid the difficulties:

$$\int \frac{(dq)}{(q^2+A)^2}\left(\frac{1}{2}q^2+A\right)=0, \tag{A}$$

$$\int \frac{(dq)}{(q^2+A)^3} A=0 \tag{B}$$

and
$$\int \frac{(dq)}{(q^2+A)^3}A^2=0. \tag{C}^*$$

These features should be taken heed of in the course of calculations of the energy-momentom tensors of elementary particles[2]. It is our purpose to re-examine the effectiveness of the above conditions on these problems.

Using the self-energy,

$$H_{self} = \int (dq) h_{self}(q^2, m), \tag{2}$$

the self-stress can be written as

$$T_{11}=\frac{1}{3}\int (dq)\frac{\partial}{\partial q_\mu}[h_{self}(q^2, m)q_\mu], \tag{3}$$

which is the equivalence formula to that obtained by Sawada.[3]

For the case of the self-stress of the fermion interacting with the neutral vector field with mass μ, using the self-energy

$$h_{self} = -\frac{ig^2}{2\pi^3}m\int_0^1 da\frac{1+a}{(q^2+L_1)^2}\bar{\phi}\phi \tag{4}$$

and

$$L_1=m^2a^2+\mu^2(1-a),$$

we obtain

$$T_{11}=-\frac{2ig^2}{3\pi^3}m\int_0^1 da(1+a)\int\frac{(dq)}{(q^2+L_1)^3}L_1\bar{\phi}\phi$$
$$=_{(B)}0$$

which is zero in virtue of condition (B). More complicated feature appears in the case of the self-stress of the vector meson interacting with the fermion, as the self-energy is

$$h_{self}=\frac{ig^2}{2\pi}\int_0^1 da\left[\frac{q^2+2L_2}{(q^2+L_2)^2}-\frac{a(a-1)\mu^2}{(q^2+L_2)^3}\right]U_\mu^2$$

and

$$L_2=m^2+\mu^2a(a-1),$$

then we have

$$T_{11}=\frac{2ig^2}{3\pi^3}\int_0^1 da\left[\int\frac{(dq)}{(q^2+L_2)^2}\left(\frac{1}{2}q^2+L_2\right)\right.$$
$$\left.+(m^2-3\mu^2a(a-1))\int\frac{(dq)}{(q^2+L_2)^3}L_2\right]U_\mu^2\underset{(A),(B)}{=}0$$

which is zero in virtue of both conditions (A) and (B). The same discussions hold in the various other types and we can get rid of these difficulties. But there is only one exception in the self-stress of the fermion interacting with the scalar field.[3] In this case we can lower the divergence from quadratic to logarithmic, but never remove it throughly. This difficulty seems to be the other one which should be attacked by the slightly different subtraction theory.

We wish to express our thanks to Mr. K. Sawada for his valuable discussions.

1) Y. Katayama, Prog. Theor. Phys. **5** (1950), 272; See also H. Fukuda and T. Kinoshita, ibid. **5** (1950), 1024.

* At first sight, the condition (C) seems not to be in agreement with F-K's condition $\int (dq)\frac{1}{2}q^2A/(q^2+A)^3=0$, but if we utilize the condition (B) simultaneously, we find the eqivalance of two conditions.

2) A. Pais and S. Epstein, Rev. Mod. Phys. **21** (1949), 445; K. Sawada, Prog. Theor. Phys. **5** (1950), 117; J. Yukawa, N Oda and H. Umezawa, ibid. **5** (1950), 320; F. Rohrlich, Phys. Rev. **77** (1950), 357; F. Villars, ibid. **79** (1950), 122.
3) K. Sawada, Prog. Theor. Phys. **5** (1950), 236.

On Yukawa's Theory of Non-local Field

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December 21, 1950

Recently Yukawa[1] discussed a possibility of generalizing the field concept by introducing a non-local field, which is free from the restriction for a field to be determined as a point function in the ordinary space. He gave an example of the non-local field, under the guiding principle of Lorentz invariance and reciprocity. The concept of non-local field being unfamiliar to us, however, we tried to investigate the relation of non-local to local field, to find these two can be transformed with each other by a canonical transformation.

The Yukawa s non-local scalar field is characterized by the equations of motion

$$\left(\frac{\partial^2}{\partial X_\mu \partial X^\mu}-\varkappa^2\right)U(X_\mu, r_\mu)=0,$$
$$(r_\mu r^\mu-\lambda^2)U(X_\mu, r_\mu)=0, \qquad (1)$$
$$r_\mu\frac{\partial U(X_\mu r_\mu)}{\partial X_\mu}=0,$$

and the commutation relation

$$[U^*(X_\mu, r_\mu),\ U(X_\mu', r_\mu')]$$
$$=\int\frac{k_4}{|k_4|}\delta(k_\mu{}^2+\varkappa^2)\delta(r_\mu{}^2-\lambda^2)\delta(k_\mu r^\mu)$$
$$\delta(r_\mu-r_\mu')e^{ik_\mu(x_\mu-x'_\mu)}\,(dk_\mu)$$

(all the notations are in accordance with Yukawa).

For our purpose it is convenient to transform (1) into a representation, in which the following three mutually commutative operators

$$\frac{\partial^2}{\partial X_\mu\partial X^\mu},\qquad r_\mu r^\mu,\qquad r_\mu\frac{\partial}{\partial X_\mu}$$

take diagonal forms. Then (1) becomes

$$(K+\varkappa^2)w(K, L, M,\cdots)=0,$$
$$(L-\lambda^2)w(K, L, M,\cdots)=0, \qquad (2)$$
$$M\,w(K, L, M,\cdots)=0,$$
$$[w^*(K, L, M, \cdots), w(K', L', M', \cdots)]$$
$$=\text{const.}\ \delta(K-K')\delta(L-L')\delta(M-M')$$
$$\times\delta(K+\varkappa^2)\delta(L-\lambda^2)\delta(M).$$

Performing a canonical transformation with a unitary operator

$$T=e^{-\lambda^2\partial/\partial L}$$

as a transformation] function, (2) is transformed into

$$(K+\varkappa^2)\overline{w}(K, L, M,\cdots)=0,$$
$$L\,\overline{w}(K, L, M,\cdots)=0,$$
$$M\,\overline{w}(K, L, M,\cdots)=0,$$
$$[\overline{w}^*(K, L, M,\cdots),\ \overline{w}(K', L', M',\cdots)]$$
$$=\text{const.}\ \delta(K-K')\delta(L-L')\delta(M-M')$$
$$\times\delta(K+\varkappa^2)\delta(L)\delta(M)$$

where

$$\overline{w}=T^{-1}wT.$$

Since λ vanishes in (3), $\overline{w}$ can be interpreted as a local field. This fact shows that the non-local and the local fields are connected by a canonical transformation. Similar trans formation is also possible in the case of spinor field. When there is interaction, Yukawa's S-matrix can be obtained from that of local field by the same canonical transformation.

Therefore nothing new is expected in Yukawa's theory in the present form. For example, the self energy of a spinor particle due to neutral non-local scalar field gives exactly the same result as that of local. To remove the divergence difficulties in the field theory, an essential revision would be necessary either in S-matrix or in the mechanism of interaction[2].

Details will soon appear in the later issue of this journal.

1) H. Yukawa, Phys, Rev. **77** (1950), 219.
2) Yennie proposed an example to modify Yukawa's theory so as to give convergent results for self-energy problems. (Phys. Rev. in press), but his procedure seems too artificial.

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